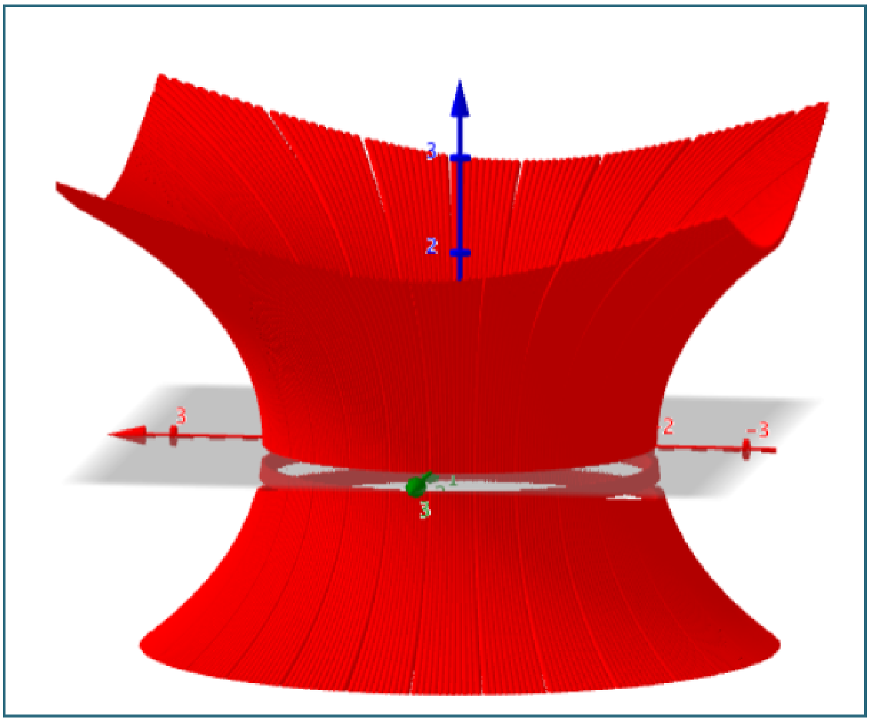
Exploration Guide



Maths IB Standard Level and Higher Level
Applications and Interpretations
Analysis and Approaches

(For first examination in 2021).

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Essential information

The exploration in IB maths has now been standardised – with the same marking criteria used for SL Analysis and SL Applications, and a slightly different marking criteria used for both HL Analysis and HL Applications.

- Your exploration can be on any area of personal interest – as long as it incorporates mathematics commensurate with the level of the course you are taking.

- Your exploration should be 12-20 pages long

- Your exploration is worth 20% of your final grade and is awarded a mark out of 20.

- During this process you should hand in a **draft version** of your exploration to your teacher. This should be what you believe to be a finished version of your exploration. Your teacher is then allowed to fully annotate this, check your mathematics and presentation before giving you written and verbal feedback as to how to improve.

- Your teacher should provide you with a copy of (or the relevant sections from):
  "Mathematics: analysis and approaches teacher support material"
  "Mathematics: analysis and approaches guide"
  Both of these two documents have detailed advice from the IB about how to choose a topic, the full marking criteria used by teachers and moderators and a lot of useful advice for completing a successful investigation.

- On the previously used similar criteria (last examination 2020) you needed the following scores for each level:

  SL: Level 7: 18-20. Level 6: 15-17. Level 5: 12-14. Level 4: 9-11.

  HL: Level 7: 17-20. Level 6: 15-16. Level 5: 12-14. Level 4: 9-11

The exploration marking criteria

You will be marked against the following criteria strands:

A: Presentation [out of 4].

B: Mathematical communication [out of 4]

C: Personal engagement [out of 3].

D: Reflection [out of 3]

E: Use of mathematics [out of 6]

A: Presentation

To achieve A4 your exploration needs to be coherent, well organised and concise.

Coherent explorations are easy to follow with steps clearly explained and sections which link together. It should be communicated so that a fellow student in your class can understand.

Organised explorations have introductions, clear aim and rationale and a conclusion. Work is cited in-text, with a bibliography. Graphs and tables are organised and presented in the relevant place.

Concise explorations have no repetitive calculations (once you have demonstrated your method once you don't need to so numerous times again). Content included should be relevant to the exploration. You should finish your exploration by achieving your aim. It should not be more than 20 pages.

Criteria A student checklist – have you done the following?

- An introduction explaining the exploration
- An aim outlining the purpose of the exploration
- A rationale explaining why the exploration was chosen
- Sensible conclusions based on the outcomes of your findings
- Discussion points and explanations that make sense to an audience of your peers
- Arguments that quickly get to the point
- Graphs &/or tables &/or diagrams &/or spreadsheets which are clearly labelled and easily understood
- Graphs with a key if needed, axes labeled appropriately
- Broken your exploration up into sections with suitable headings
- Included page numbers and bibliography

B: Mathematical Communication

To achieve B4 your mathematical communication must be relevant, appropriate and consistent throughout.

Appropriate means that the mathematical presentation will be as required. This means:

a) Good use of an equation editor if you use any equations. No computer notation.

b) Having all tables clearly labelled.

c) Having all graphs with a title and with axes labelled.

d) Having defined all variables and explained all mathematical terms.

e) Rounding to an appropriate degree, and demonstrating an understanding of the level of accuracy you are using.

Relevant means that you choose the mathematical presentation that is required to communicate the topic aim. Including graphs or tables which are not necessary in addressing your aim is not relevant.

Consistent means that your mathematical presentation does not change throughout the exploration (e.g. changing from y to Y) and is consistently appropriate.

Criteria B student checklist – have you done the following?

- Used correct mathematical language throughout the exploration
- Used correct mathematical notation and symbols throughout the exploration – **no** computer or GDC notation is acceptable ie *, ^, /
- Used the approximately equal to sign when rounding numbers
- Defined terms and variables **before** they are used in your working and referred to in your explanations
- Used multiple forms of mathematical representation such as formulae, diagrams, tables, charts, graphs and models as appropriate

C: Personal Engagement

To achieve C3 there must be evidence of outstanding personal engagement.
You can show **significant** personal engagement [C2] by:

a) Thinking independently – looking at a problem from a unique perspective. Very formulaic investigations or ones which are very similar to textbook problems are to be avoided.

b) Showing your personal interest – explain why you are so interested in this exploration and maintain your personal voice throughout when commenting, explaining and reflecting.

c) Exploring the topic – this is an exploration and not a summary of other people's work. Try to explore from different perspectives or using different mathematical methods.

d) Trying to learn some new mathematics or trying to apply mathematics you already know to new contexts.

e) If you are doing an investigation which requires data then try to collect this data yourself. Show how you were engaged in this process.

You can show **abundant** personal engagement [C3] by doing a significant amount of the ideas expressed above, to a high quality and showing a creative approach to meeting the aim of the exploration.

Criteria C student checklist – have you done the following?

- Clearly outlined or identified a personal connection with your exploration in your rationale with reasons
- Researched relevant background information for you exploration
- Related the exploration to your personal experiences
- Used self-generated data if possible
- Related back to your interest shown in the introduction
- Drawn conclusions, made reflections and arguments that show originality of thought
- Used Maths that you have learnt yourself, or used and applied mathematics from the syllabus in a new context.
- Made suitable improvements using the feedback given for the draft version of the exploration.

D: Reflection

To achieve D3 you must show substantial evidence of critical reflection.

You can show **meaningful reflection** [D2] by:

a) Commenting on what you have learnt and how it links to your initial aim.

b) Providing reflection on results – put them into real life context where possible. What are the implications of your results?

c) Comparing different approaches to solving a problem and addressing the accuracy of these methods.

d) Looking at the strengths, weaknesses and limitations of your investigation and providing ideas for further study.

To show **substantial critical reflection** [D3] you should be doing all of the above, to a very high standard and doing it throughout the exploration.

Criteria D student checklist – have you done the following?

Have you

- Made comments on the results **and** the process of the exploration
- Made comments on how the results and the process could be improved
- Made comments within the context of the subject matter of the exploration
- Made sensible recommendations based on your findings
- Drawn conclusions that refer back to the aim and rationale of your exploration
- Made comments on possible limitations of your exploration
- Considered and discussed future implications of the results of the exploration that could perhaps lead to further explorations

E: Use of Mathematics

Standard Level students:

To achieve at least E3 the mathematics must be **commensurate** with the level of the Standard Level course. This means it should be using mathematics from the Standard Level syllabus or mathematics of a similar level. The maths used should be **relevant** to the stated aim of the investigation. If complicated maths is used when much simpler maths could have been used instead then this is not relevant.

To achieve at least E5 students need to **demonstrate** good knowledge and understanding. The IB defines demonstrate as, "to make clear by reasoning or evidence, illustrating with examples or practical application." Each step of any calculation should be clearly explained and any thought processes written down to demonstrate that the student does clearly understand the mathematics they are using.

To achieve E6 students need for all the mathematics to be **correct** and to have demonstrated a **thorough knowledge** and understanding of their topic. Students who choose an exploration that goes beyond the scope of SL maths usually struggle to demonstrate thorough understanding (and often good understanding).

Higher Level students:

To achieve E3 the mathematics used by HL students must be **correct.** It must also be **relevant** to the exploration and **commensurate** with the level of the course. [contained within the syllabus, see above]

To achieve E5 students must demonstrate **sophistication** or **rigour** and **thorough** knowledge.

Sophisticated mathematics should be that which is specifically on the HL rather than SL syllabus, or SL maths which has been extended beyond what an SL student would be reasonably expected to produce.

Rigour requires a high level of logical thought in all calculations and justification for claims made (or proof presented).

To achieve E6 students must show both sophistication **and** rigour and also **precise** mathematics – every calculation must be correct and relevant levels of accuracy considered throughout.

Summary student checklist for submissions

Criteria A – Presentation

Have you

- An introduction explaining the exploration
- An aim outlining the purpose of the exploration
- A rationale explaining why the exploration was chosen
- Sensible conclusions based on the outcomes of your findings
- Discussion points and explanations that make sense to an audience of your peers
- Arguments that **quickly** get to the point
- Graphs &/or tables &/or diagrams &/or spreadsheets which are clearly labelled and easily understood
- Graphs with a key if needed, axes labeled appropriately
- Broken your exploration up into sections with suitable headings
- Included page numbers
- Included a bibliography

Criteria B – Mathematical Communication

Have you

- Used correct mathematical language throughout the exploration
- Used correct mathematical notation and symbols throughout the exploration – **no** computer or GDC notation is acceptable ie *, ^, /
- Used the approximately equal to sign when rounding numbers
- Defined terms and variables **before** they are used in your working and referred to in your explanations
- Used multiple forms of mathematical representation such as formulae, diagrams, tables, charts, graphs and models as appropriate

Criteria C – Personal Engagement

Have you

- Clearly outlined or identified a personal connection with your exploration in your rationale with reasons
- Researched relevant background information for you exploration using external sources
- Related the exploration to your personal experiences
- Used self-generated data if possible
- Drawn conclusions that relate back to the original rationale and your interest shown in the introduction
- Drawn conclusions, made reflections and arguments that show originality of thought
- Used Maths that goes beyond the level of the course with understanding
- Made suitable improvements using the feedback given for the draft version of the exploration

Criteria D – Reflection

Have you

- Made comments on the results **and** the process of the exploration
- Made comments on how the results and the process could be improved
- Made comments within the context of the subject matter of the exploration
- Made sensible recommendations based on your findings
- Drawn conclusions that refer back to the aim and rationale of your exploration
- Made comments on possible limitations of your exploration
- Considered and discussed future implications of the results of the exploration that could perhaps lead to further explorations

Criteria E – Use of Mathematics

Have you

- Used relevant mathematics commensurate with the level of the course
- Made sure all the mathematics used is correct
- Demonstrated a thorough knowledge and understanding of the mathematics

Choosing a topic

1) Choose something you are interested in finding out more about. What course do you want to do at university? What are your other IB subjects – could you link with them? What are your hobbies and interests? But remember this is a **maths investigation** – the topic you choose has to be of a level relevant to your IB Maths course.

2) Choose a topic that you are able to explore – and ideally that has a question you can seek the answer to. Do not simply produce a summary of some maths. You need to be able to look for an answer to a question – which then can allow you to reflect on later.

3) Choose a topic that is at the right level of maths for the course. Too easy and you will not get a good grade on criteria E, too hard and you could lose marks on criteria A and E.

4) Try to avoid topics that have been done hundreds of times before – you will find it much easier to impress the moderator with an interesting and novel investigation rather than one he/she has seen many times before.

Main Topic areas

1) Modelling functions

2) Pure Maths

3) Geometry

4) Stats and probability

5) Calculus

6) Linking with other subjects

Modelling functions

The main idea

Fit a curve to a function or describe a real life process through equations.

Why is this topic a good idea?

This topic can allow you to link functions, transforming graphs and calculus together.

Examples:

1) Traffic flow: How maths can model traffic on the roads.

2) Impact Earth – what would happen if an asteroid or meteorite hit the Earth?

3) Modelling infectious diseases – how we can use mathematics to predict how diseases like measles will spread through a population

4) Modelling Zombies – How do zombies spread? What is your best way of surviving the zombie apocalypse? Surprisingly maths can help!

5) Modelling music with sine waves – how we can understand different notes by sine waves of different frequencies. Listen to the sounds that different sine waves make.

6) The Gini Coefficient – How to model economic inequality

7) Maths of Global Warming – Modeling Climate Change – Using Desmos to model the change in atmospheric Carbon Dioxide.

8) Modelling radioactive decay – the mathematics behind radioactivity decay, used extensively in science.

9) Torus – solid of revolution: A torus is a donut shape which introduces some interesting topological ideas.

10) Projectile motion: Studying the motion of projectiles like cannon balls is an essential part of the mathematics of war.

11) Batman and Superman maths – how to use Wolfram Alpha to plot graphs.

Pure Maths

The main idea:

Pure maths is often involved in finding patterns and rules that help us understand sequences and numbers.

Why is this topic a good idea?

This topic allows you to experience what "real" mathematicians do with their time - take a problem and then explore how it can be solved. Can it be solved using different methods? Which method is best?

Examples:

1) Plotting the Mandelbrot set: The stunning graphics of Mandelbrot and Julia Sets are generated by complex numbers.

2) Waging war with maths - an investigation into hollow squares and hollow cubes.

3) The Mordell equation: How to find the differences between square and cube numbers.

4) Ramanujan's taxi cab number - explore the links between number theory, graphs and group theory.

5) Chinese remainder theorem. This is a puzzle that was posed over 1500 years ago by a Chinese mathematician. It involves understanding the modulo operation.

6) Perfect Numbers: Perfect numbers are the sum of their factors (apart from the last factor). ie 6 is a perfect number because 1 + 2 + 3 = 6.

7) Time travel to the future: Investigate how traveling close to the speed of light allows people to travel "forward" in time relative to someone on Earth.

8) RSA code – the most important code in the world? How all our digital communications are kept safe through the properties of primes.

9) The Chinese Remainder Theorem: This is a method developed by a Chinese mathematician Sun Zi over 1500 years ago to solve a numerical puzzle.

10) Square triangular numbers - How to use number theory and computing to find numbers which are both square and triangular numbers.

Geometry

The main idea:

Geometry looks at some form of describing images through lines and points. A part of mathematics over 2000 years old.

Why is this topic a good idea?

This topic allows you to combine investigation techniques alongside graphical software.

Examples:

1) Non-Euclidean geometries: This allows us to "break" the rules of conventional geometry – for example, angles in a triangle no longer add up to 180 degrees.

2) Mandelbrot set and fractal shapes: Explore the world of infinitely generated pictures and fractional dimensions. Also Sierpinski triangles: a fractal design.

3) Soap bubbles, wormholes and catenoids - explore minimal surfaces and the shapes they create.

4) Graphically understanding complex roots – have you ever wondered what the complex root of a quadratic actually means graphically?

5) Circular inversion – what does it mean to reflect in a circle? A great introduction to some of the ideas behind non-euclidean geometry.

6) Graphing Stewie from Family Guy. How to use graphic software to make art from equations.

7) The Coastline Paradox – how we can measure the lengths of coastlines, and uses the idea of fractals to arrive at fractional dimensions.

8) The Folium of Descartes. This is a nice way to link some maths history with studying an interesting function.

9) Measuring the Distance to the Stars. Maths is closely connected with astronomy – see how we can work out the distance to the stars.

10) Euler's 9 Point Circle. This is a lovely construction using just compasses and a ruler.

Stats and Probability 1

The main idea:

Stats and probability help us understand a variety of other subjects and are useful for predictions. These topics below have scope for in-depth investigation.

Why is this topic a good idea?

This allows you to link mathematical techniques with other subjects in the sciences.

Examples:

1) Quantum mechanics - statistical universe._ Look at how probability underpins reality.

2) The Martingale system - investigate how a centuries old system of doubling down on losing bets is still used in currency trading today.

3) Using Chi Squared to crack codes – Chi squared can be used to crack Vigenere codes which for hundreds of years were thought to be unbreakable. Unleash your inner spy!

4) Are you psychic? Use the binomial distribution to test your ESP abilities.

5) Reaction times – are you above or below average? Model your data using a normal distribution.

6) Modelling volcanoes – look at how the Poisson distribution can predict volcanic eruptions, and perhaps explore some more advanced statistical tests.

7) Could Trump win the next election? How the normal distribution is used to predict elections.

8) Gambler's fallacy: A good chance to investigate misconceptions in probability and probabilities in gambling. Why does the house always win?

9) Birthday paradox: The birthday paradox shows how intuitive ideas on probability can often be wrong. How many people need to be in a room for it to be at least 50% likely that two people will share the same birthday? Find out!

Stats and Probability 2

The main idea:

Stats and probability help us understand a variety of other subjects and are useful for predictions. These topics below lend themselves to simpler stats investigations.

Why is this topic a good idea?

This allows you to link mathematical techniques with other subjects in the sciences

1) Is there a correlation between the digit ratio and maths ability? Studies suggest there is a correlation between digit ratio and academic ability. Is this true?

2) Is there a correlation between GDP and life expectancy? Run the Gapminder graph to show the changing relationship between GDP and life expectancy over the past few decades.

3) Is there a correlation between stock prices of different companies? Use Google Finance to collect data on company share prices.

4) Is there a correlation between Premier League wages and league positions?

5) Are the IB maths test scores normally distributed? IB test scores approximately fit bell curves. Investigate how the scores from different IB subjects compare.

6) Investigation into the distribution of word lengths in different languages. The English language has an average word length of 5.1 words. How does that compare with other languages?

7) Do bilingual students have a greater memory recall than non-bilingual students? Studies have shown that bilingual students have better "working memory" – does this include memory recall?

8) Are a sample of student heights normally distributed? We know that adult population heights are normally distributed – what about student heights?

9) Which times tables do students find most difficult to learn? – Are some calculations like 7×8 harder than others? Why?

Calculus

The main idea:

Calculus is a very powerful tool for understanding the world around us

Why is this topic a good idea?

Allows you to look at a mixture of modelling and pure maths together and to use a number of calculus techniques.

Examples:

1) The Monkey and the Hunter – How to Shoot a Monkey – Using Newtonian mathematics to decide where to aim when shooting a monkey in a tree.

2) How to Design a Parachute – looking at the physics behind parachute design to ensure a safe landing!

3) Galileo: Throwing cannonballs off The Leaning Tower of Pisa – Recreating Galileo's classic experiment, and using maths to understand the surprising result.

4) Area optimisation_ – an investigation.

5) Bullet projectile motion experiment – using Tracker software to model the motion of a bullet.

6) Radiocarbon dating – understanding radioactive decay allows scientists and historians to accurately work out something's age – whether it be from thousands or even millions of years ago.

7) Intergalactic space travel and time dilation – Essential knowledge for future astronauts.

8) Lagrange points – and how these are used for satellites. Investigate some rocket science!

9) Creating 3D shapes using volume of revolutions - volume of a rugby ball.

10) Volume optimisation of cuboids - an investigation.

Linking with other subjects and areas in maths

The main idea:

Maths is used across many different areas – explore some of these.

Why is this topic a good idea?

Allows you to look at another subject using mathematical ideas. Good if you like another subject (perhaps if you want to do this at university).

Examples:

1) Mathematical methods in economics – maths is essential in both business and economics – explore some economics based maths problems.

2) Genetics – Look at the mathematics behind genetic inheritance and natural selection.

3) Medical data mining – Explore the use and misuse of statistics in medicine and science.

4) Designing bridges – Mathematics is essential for engineers such as bridge builders – investigate how to design structures that carry weight without collapse.

5) Solving maths problems using computers – computers are really useful in solving mathematical problems. Here are some examples solved using Python.

6) Simulating a football season using maths.

7) Codes and ciphers: ISBN codes and credit card codes are just some examples of how codes are essential to modern life. Maths can be used to both make these codes and break them.

8) The Tusi circle: Circles rolling inside circles using parametric equations.

9) Telephone numbers - explore some graph theory and how it can link to calculus.

Useful programs/sites

1) If you are interested in the environment, The National Oceanic and Atmospheric Administration (NOAA) has collected a large amount of environmental data for use.

2) If you like football you can also find a lot of football stats on the Who Scored website. This gives you data on things like individual players' shots per game, pass completion rate etc.

3) The **World Bank** has a huge data bank – which you can search by country or by specific topic. You can compare life-expectancy rates, GDP, access to secondary education, spending on military, social inequality, how many cars per 1000 people and much much more.

4) Gapminder is another great resource for comparing development indicators – you can plot 2 variables on a graph (for example urbanisation against unemployment, or murder rates against urbanisation) and then run them over a number of years. You can also download Excel spreadsheets of the associated data.

5) Wolfram Alpha is one of the most powerful maths and statistics tools available – it has a staggering amount of information that you can use.

6) Plotly is a great visual graphic site – you can create visually interesting infographics and analyse data from hundreds of other sources.

7) TSM – the Technology for Secondary Mathematics is something of an internet dinosaur – but has a great deal of downloadable data files on everything from belly-button ratios to lottery number analysis and baby weights.

8) Google Public Data – an enormous source for public data, which is displayed graphically and can be searched.

9) Nationmaster – another huge site with pretty much any statistic and data comparing countries. Currently they have 19 million data points!

10) Desmos – a great online graphing site

11) Geogebra – another very powerful graphing application (also does 3D)

12) Tracker software will allow you to track data from a video and then produce graphs.

Common mistakes

A: Communication strand.

You should have a clear aim which allows you to investigate in a logical manner and then reflect and conclude by completing this aim.

"My aim is to investigate projectiles in football" is a **poor aim**. It is vague and doesn't naturally allow a good investigative flow.

"My aim is to compare Ronaldo's free-kicks with my free-kicks to become a better footballer" is a **much better aim**. This investigation will have a clear flow, lots of opportunities for personal engagement, reflection and could be successfully completed.

C: Personal engagement strand.

Choose a topic which allows you to express yourself personally. Whilst other students may have done the same topic you, yours should still be sufficiently personalised to be unique.

A topic which will **struggle** to show good personal engagement:

"I will look at the correlation between GDP and test scores for 10 countries."

[Data will be secondary, difficult to show significant personal angle, basic correlation investigations are very common and formulaic].

A topic which **could show** good personal engagement:

"Designing a children's swing bridge for our school's playground"

[Opportunity for personal data collection, opportunity to produce a unique investigation, opportunity for use of some self-taught maths, opportunity for links with DT in making a model].

B: Mathematical Communication strand.

Example 1. Poor communication:

f(x) = 3x/2 +4*sinx
When X = 1, I get 4.9 metres.

Example 1. Good communication:

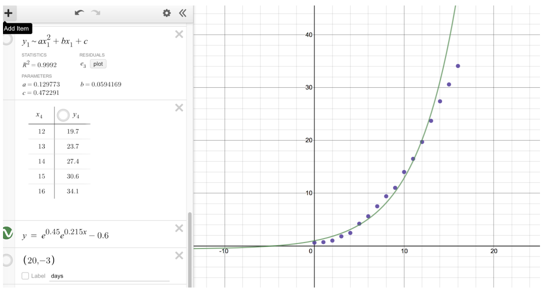
t is the time in seconds. *h(t)* is the height in metres. Angles are in radians.

$$h(t) = \frac{3t}{2} + 4sin(t)$$

$$h(1) = \frac{3(1)}{2} + 4sin(1)$$

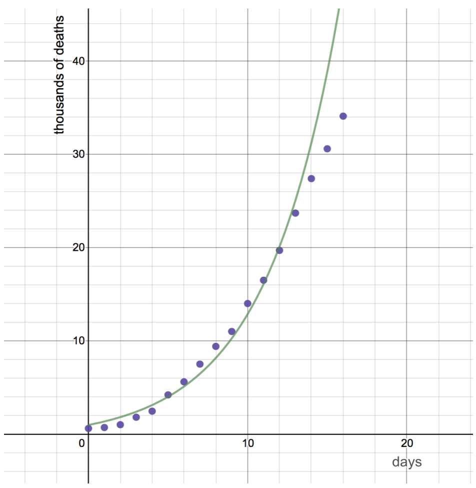
$h(1) \approx$ 4.87 (to 3 significant figures).

Example 2. Poor communication: Poor screen capture, hard to read, axes not labelled, no title



Example 2. Good communication: [axes labelled, graph given a title, domain restricted]

Figure 1: Plotting the number of deaths in a Covid-19 outbreak



D: Reflection strand.

Example 1. Poor reflection:

I did linear regression and got an r value of 0.91. This shows there is strong positive correlation.

Example 1. Good meaningful reflection:

I did linear regression and got an r value of 0.91. This shows there is a strong positive correlation between height and weight. As weight is my dependent variable this shows that as someone's height increases, their weight also increases. My linear regression line of best fit is:

$$w = 10h + 40$$

w : *weight in kg*
h : *height in metres*

This means that for every extra metre in height we would expect someone to weigh an extra 10kg. However this equation is only valid for male adults, and because of the data we used the domain should be restricted to $h \geq 1.5$. We can see that without restricting the domain we would have a prediction of someone with no height being 40kg. This clearly makes no sense. I can use my equation to predict the weight of someone who is 1.7m tall:

$$w = 10(1.7) + 40$$
$$w = 57$$

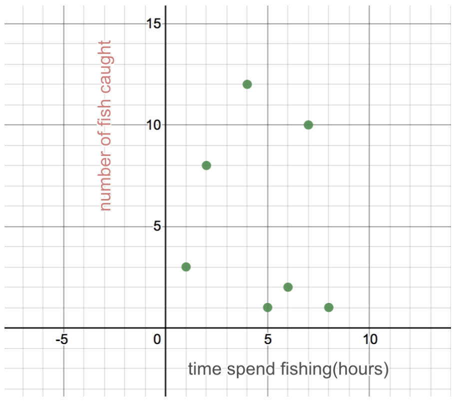
This seems like a reasonable prediction because I measured my classmate who is 1.7m tall and their weight was 63kg. This gives me confidence that my model is accurate.

E: Use of mathematics strand.

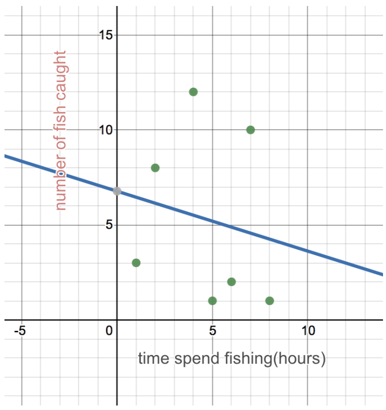
For SL students I'd strongly advise you not to choose mathematics that is more difficult than the level of the SL course. Students rarely are able to demonstrate good understanding and hence get a poor mark. HL students can potentially go a bit beyond the HL syllabus in terms of difficulty level - but it should not be much beyond.

Statistics projects frequently perform poorly in the E category - to demonstrate good understanding you need to show you understand why you're using the test you're using, what assumptions this test relies on, your sampling considerations etc. The most common low scoring stats investigation is a simple correlation between 2 variables where students first draw a scatter graph, note that there is weak or no correlation and then do a Pearson's product calculation. If the scatter graph already shows weak or no correlation then it is not relevant to do this.

Example 1. Poor mathematical understanding:



I plotted the number of fish caught and time spent fishing. From my scatter graph I can see there is no correlation or very weak correlation. I then used Pearson's Product formula to find that $r \approx -0.175$ *(3sf).* I then found the line of best fit:

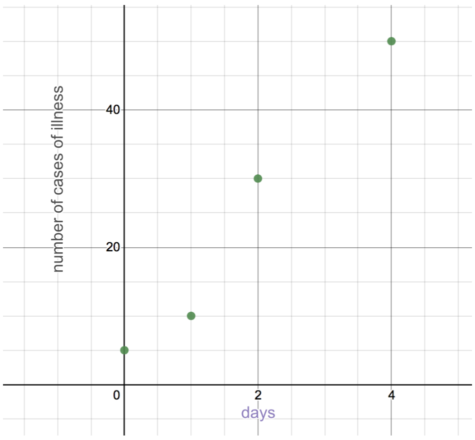


I will now use this line to predict how many fish will be caught after 15 hours.

Why does this show poor mathematical understanding?

Firstly there are only 7 data points - so any conclusions drawn from this are going to be weak at best. The student draws a scatter graph [this will achieve E2 for relevant maths], but because the scatter graph shows weak or no correlation then the following maths to use the Pearson's Product formula is not relevant. It is also not good understanding to use a line of best fit for such poorly fitting data, and not relevant in this case to extrapolate beyond the data range as this model suggests the more time you spend fishing the less you will catch!

Example 2. Poor mathematical understanding:



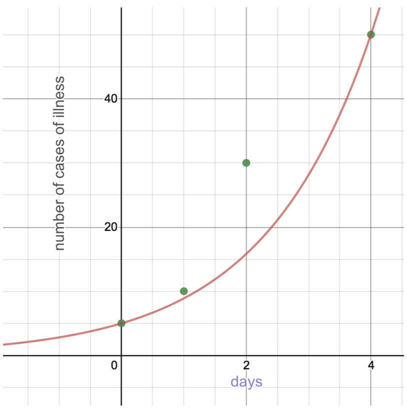
I plotted the graph above and tried to fit an exponential curve to the data points. The graph needs to pass through (0,5) and (4,50), so I can make the following equations:

$5 \ = ae^{b(0)}$

$50 \ = ae^{b(4)}$

Solving these give:

y = $5e^{0.576x}$



I also did an exponential regression using desmos to check the validity of an exponential model and got:

y = $9.39e^{0.425x}$

This gave an R^2 value of 0.912. This is close to 1 which shows that an exponential model is valid.

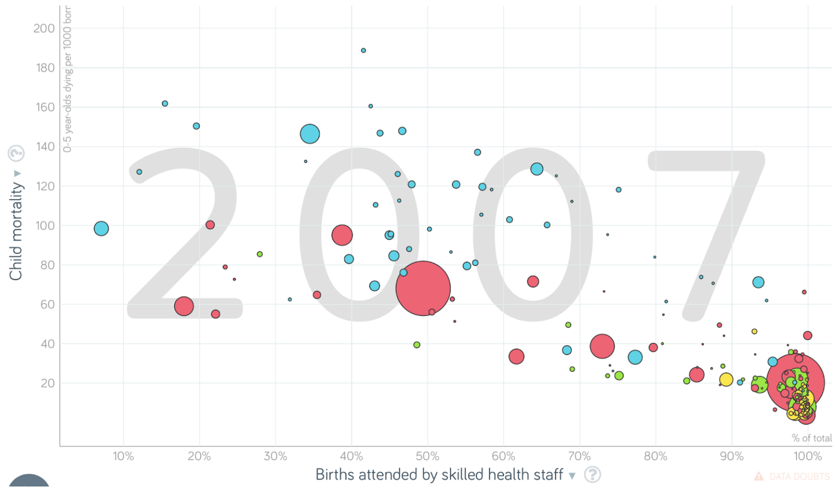
Why does this show poor mathematical understanding?
There are only 4 data points to use, the student chooses the first and last data point without justification. The exponential model is also chosen without justification - when clearly a linear model would be a better fit. The student does not show their method (and hence does not show understanding) of how to solve the simultaneous equations. R^2 is not explained and does not give evidence that an exponential model is valid for this data.

Advice for statistical explorations

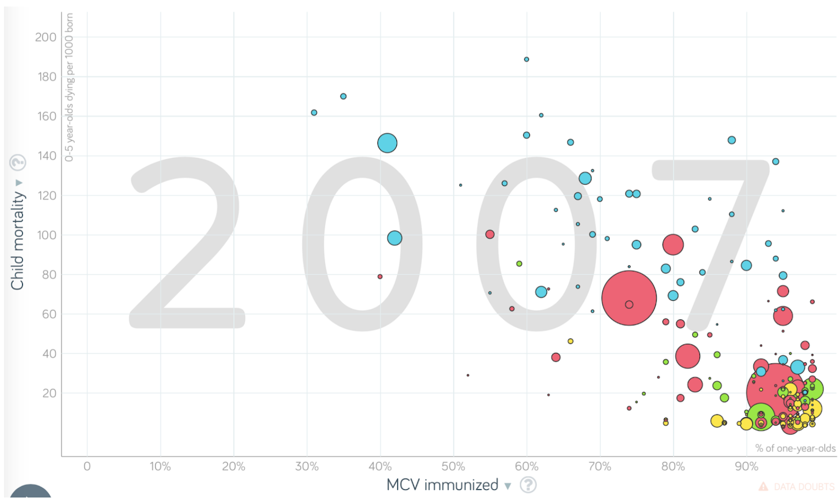
A large number of students choose to do statistical investigations. As such the IB have previously given some excellent advice (aimed at the old Maths Studies course) but very useful for the new syllabus as well. I will summarise that advice below and add my own thoughts.

1) Because statistical investigations are so common you want to try and stand out from the crowd. One interesting way of doing this is to compare the effect of 2 independent variables on a third. For example if I am working in a public health role with a goal of reducing child mortality I might want to look at whether child mortality is dependent on births attended by skilled health staff and whether it is dependent on measles vaccination. I could then find the correlation coefficient for both graphs to inform a policy as to which area I was to launch a new initiative to fund.

Is child mortality dependent on births attended by skilled health staff?

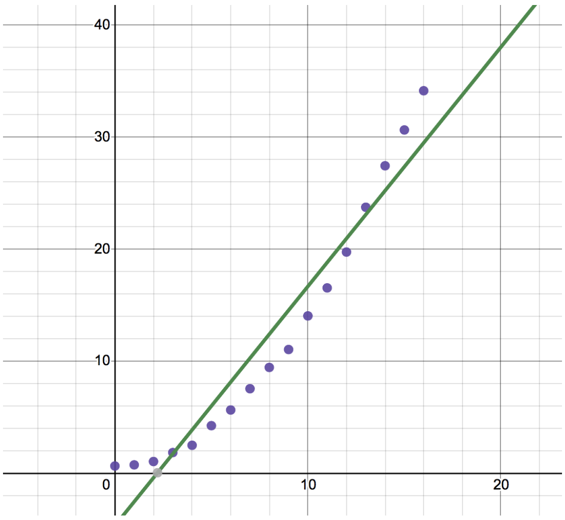


Is child mortality dependent on measles vaccination?

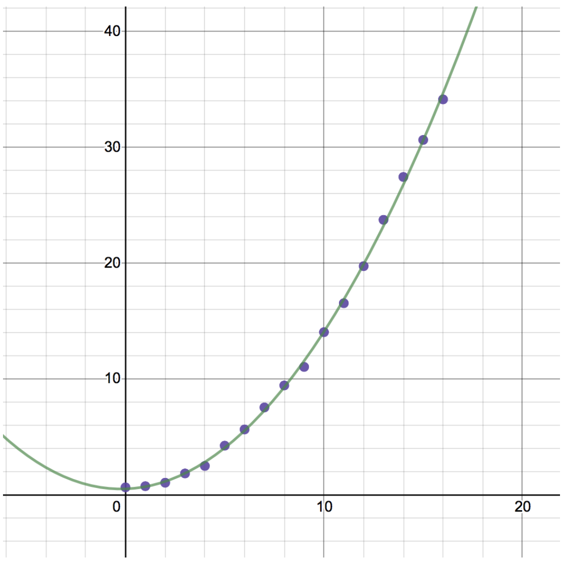


In these sorts of comparisons it's quite difficult to find 2 independent variables to compare - for example measles vaccinations will likely be higher in countries with more advanced health care systems which provide skilled professionals to attend births. Nevertheless we have a lot more interesting areas to discuss and explore than a simplistic correlation investigation.

2) If you have considered a linear model and found reasonably high correlation you could then compare this linear model to other models using technology. For example the follow graph gives a linear regression *r* value of 0.966:



But really a linear model is not the most appropriate for this data. Instead if we use a quadratic model we can see that this is a much better fit:



So don't just assume that a linear model is most appropriate because you get an *r* value close to 1 or -1.

Examples of interesting topics to explore.

I have included 6 examples of exploration topics which provide plenty of scope for interesting investigations. These were written as website posts and so are not intended to be exploration exemplars - merely to show some possible ideas for what can be done. For example computer notation (such as /) is used, not all axes are labelled, they would need more reflection etc.

1. **Calculus: Envelope of projectile motion [Suitable for HL students]**

Why is this an interesting topic? This allows you to work with generalised functions, and go beyond usual projectile motion explorations.

2. **Number sequences and proof: Square Triangular Numbers [suitable for HL students]**

Why is this an interesting topic? This allows you to combine computing and classic proof from number theory.

3. **Modelling and Calculus: Finding the volume of a rugby ball (prolate spheroid). [Suitable for SL students]**

Why is this an interesting topic? This allows you to explore a real life use of calculus.

4. **Trigonometry and modelling. Tides: What is the effect of a full moon? [Suitable for SL students]**

Why is this an interesting topic? This allows you to explore a real life use of modelling and trigonometry combined with technology.

5. **Modelling. Plotting Stewie Griffin from Family Guy [Suitable for SL students]**

Why is this an interesting topic? This allows you to explore the power of graphical software and combine mathematics with art and design.

6. **Calculus optimisation. Volume optimization of a cuboid [Suitable for HL students]**

Why is this an interesting topic? This allows you to generalise optimisation methods and apply calculus to a real life setting.

1. Calculus: Envelope of projectile motion [Suitable for HL]

For any given launch angle and for a fixed initial velocity we will get projectile motion. In the graph above I have changed the launch angle to generate different quadratics. The black dotted line is then called the envelope of all these lines, and is the boundary line formed when I plot quadratics for every possible angle between 0 and pi.

Finding the equation of an envelope for projectile motion

Let's start with the equations for projectile motion, usually given in parametric form:

$$x = (v\cos\theta)t$$

$$y = (v\sin\theta)t - 0.5gt^2$$

Here v is the initial velocity which we will keep constant, theta is the angle of launch which we will vary, and g is the gravitational constant which we will take as 9.81.

First let's rearrange these equations to eliminate the parameter t.

$$\frac{x}{v\cos\theta} = t$$

$$y = (v\sin\theta)\frac{x}{v\cos\theta} - 0.5g(\frac{x}{v\cos\theta})^2$$

$$y = x\tan\theta - 0.5g\frac{x^2}{v^2\cos^2\theta}$$

Next, we use the fact that the envelope of a curve is given by the points which satisfy the following 2 equations:

$$F(x, y, \theta) = 0$$

$$\frac{\partial F}{\partial \theta}(x, y, \theta) = 0$$

F(x,y,theta)=0 simply means we have rearranged an equation so that we have 3 variables on one side and have made this equal to 0. The second of these equations means the partial derivative of F with respect to theta. This means that we differentiate as usual with regards to theta, but treat x and y like constants.

Therefore we can rearrange our equation for y to give:

$$F(x, y, \theta) = xtan\theta - 0.5g\,\frac{x^2}{v^2cos^2\theta} - y = 0$$

and in order to help find the partial differential of F we can write:

$$F(x, y, \theta) = xtan\theta - (\frac{gx^2}{2v^2})cos^{-2}\theta - y = 0$$

$$\frac{\partial F}{\partial \theta}(x, y, \theta) = xsec^2\theta + 2(\frac{gx^2}{2v^2})cos^{-3}\theta(-sin\theta) = 0$$

$$\frac{\partial F}{\partial \theta}(x, y, \theta) = \frac{x}{cos^2\theta} - 2(\frac{gx^2}{2v^2})\frac{tan\theta}{cos^2\theta} = 0$$

$$\frac{\partial F}{\partial \theta}(x, y, \theta) = x - (\frac{gx^2}{v^2})tan\theta = 0$$

We can then rearrange this to get x in terms of theta:

$$tan\theta = \frac{v^2}{xg}$$

$$\theta = arctan\,\frac{v^2}{xg}$$

We can then substitute this into the equation for F(x,y,theta)=0 to eliminate theta:

$$F(x, y, \theta) = xtan\theta - 0.5g\,\frac{x^2}{v^2cos^2\theta} - y = 0$$

$$F(x, y, \theta) = x\frac{v^2}{xg} - 0.5g\,\frac{x^2}{v^2cos^2arctan\,\frac{v^2}{xg}} - y = 0$$

We then have the difficulty of simplifying the second denominator, but luckily we have a trig equation to help:

$$cosarctan\theta = \frac{1}{\sqrt{1+\theta^2}}$$

Therefore we can simplify as follows:

$$cos^2arctan\theta = \frac{1}{1+\theta^2}$$

$$cos^2arctan\frac{v^2}{xg} = \frac{1}{1+\frac{v^4}{x^2g^2}}$$

$$cos^{-2}arctan\frac{v^2}{xg} = 1+\frac{v^4}{x^2g^2}$$

and so:

$$F(x, y, \theta) = x\frac{v^2}{xg} - 0.5g\,\frac{x^2}{v^2cos^2arctan\,\frac{v^2}{xg}} - y = 0$$

$$F(x, y, \theta) = \frac{v^2}{g} - 0.5g\frac{x^2}{v^2}\,(1+\frac{v^4}{x^2g^2}) - y = 0$$

$$F(x, y, \theta) = \frac{v^2}{g} - 0.5g\frac{x^2}{v^2} - 0.5\frac{v^2}{g} - y = 0$$

$$F(x, y, \theta) = \frac{v^2}{2g} - \frac{gx^2}{2v^2} - y = 0$$

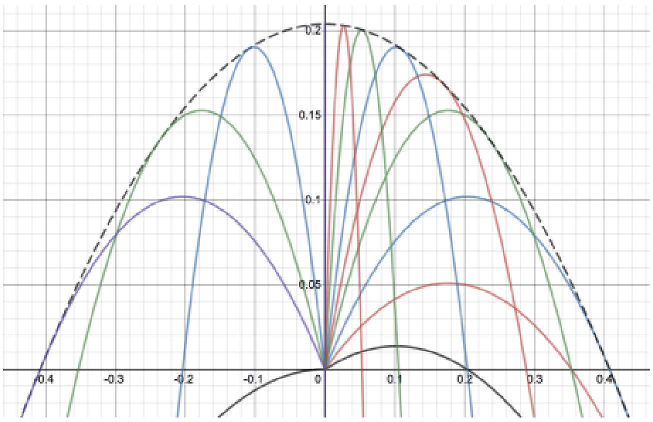
$$\frac{v^2}{2g} - \frac{gx^2}{2v^2} = y$$

And we have our equation for the envelope of projectile motion! As we can see it is itself a quadratic equation. Let's look at some of the envelopes it will create. For example if I launch a projectile with a velocity of 2, and taking g = 9.81, I get:

$$\frac{4}{2(9.81)} - \frac{9.81x^2}{8} = y$$

$$x = (2cos\theta)t$$

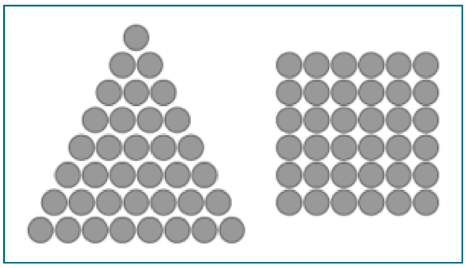
$$y = (2sin\theta)t - 0.5(9.81)t^2$$



This is the envelope of projectile motion when I take the following projectiles in parametric form and vary theta from 0 to pi:

So, there we have it, we can now create the equation of the envelope of curves created by projectile motion for any given initial velocity!

2. Number sequences and proof: Square Triangular Numbers [suitable for HL students]



Square triangular numbers are numbers which are both square numbers and also triangular numbers – i.e they can be arranged in a square or a triangle. The picture above (source: wikipedia) shows that 36 is both a square number and also a triangular number. The question is how many other square triangular numbers we can find?

The equation we are trying to solve is:

$a^2 = 0.5(b^2+b)$

for some a, b as positive integers. The LHS is the formula to generate square numbers and the RHS is the formula to generate the triangular numbers.

We can start with some simple Python code (which you can run here):

```
for c in range(1,10001):
 for d in range(1,10001):
  if c**2 == (d**2+d)/2:
   print(c**2, c,d)
```

This checks the first 10000 square numbers and the first 10000 triangular numbers and returns the following:

1 1 1

36 6 8

1225 35 49

41616 204 288

1413721 1189 1681

48024900 6930 9800

i.e 1225 is the next square triangular number after 36, and can be formed as 35^2or as $0.5(49^2+49)$. We can see that there are very few square triangular numbers to be found in the first 50 million numbers. The largest we found was 48,024,900 which is made by 6930^2 or as $0.5(9800^2+9800)$.

We can notice that the ratio between each consecutive pair of square triangular numbers looks like it converges as it gives:

36 divided by 1 = 36

1225 divided by 36 = 34.027778

41616 divided by 1225 = 33.972245

1413721 divided by 41616 = 33.970612

48024900 divided by 1413721 = 33.970564

So, let's use this to predict that the next square triangular number will be around

48024900 x 33.9706 = 1,631,434,668.

If we square root this answer we get approximately 40391

If we solve $0.5(b^2+b) = 1,631,434,668$ using Wolfram we get approximately 57120.

Therefore let's amend our code to look in this region:

```
for c in range(40380,40400):
 for d in range(57100,57130):
  if c**2 == (d**2+d)/2:
   print(c**2, c,d)
```

This very quickly finds the next solution as:

1631432881 40391 57121

This is indeed 40391^2 – so our approximation was very accurate. We can see that this also gives a ratio of 1631432881 divided by 48024900 = 33.97056279 which we can then use to predict that the next term will be 33.970563 x 1631432881 = 55,420,693,460. Square rooting this gives a prediction that we will use the 235,416 square number. $235{,}416^2$ gives 55,420,693,056 (using Wolfram Alpha) and this is indeed the next square triangular number.

So, using a mixture of computer code and some pattern exploration we have found a method for finding the next square triangular numbers. Clearly we will quickly get some very large numbers – but as long as we have the computational power, this method should continue to work.

Using number theory

The ever industrious Euler actually found a formula for square triangular numbers in 1778 – a very long time before computers and calculators, so let's have a look at his method:

We start with the initial problem, and our initial goal is to rearrange it into the following form:

$$a^2 = \frac{1}{2}(b^2 + b)$$

$$2a^2 = b^2 + b$$

$$2a^2 = (b + 0.5)^2 - 0.25$$

$$2a^2 + 0.25 = (b + 0.5)^2$$

$$8a^2 + 1 = 4(b + 0.5)^2$$

$$8a^2 + 1 = 2^2(b + 0.5)^2$$

$$8a^2 + 1 = (2(b + 0.5))^2$$

$$8a^2 + 1 = (2b + 1)^2$$

Next we make a substitution:

$$x = 2b + 1, \qquad y = 2a, \qquad y^2 = 4a^2$$

$$2y^2 + 1 = x^2$$

$$1 = x^2 - 2y^2$$

$$x = P_{2k} + P_{2k-1}$$

$$y = P_{2k}$$

Here, when we get to the equation 1 = x^2 – $2y^2$ we have arrived at a Pell Equation (hence the rearrangement to get to this point). This particular Pell Equation has the solution quoted above where we can define P_k as

$$P_k = \frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}}$$

Therefore we have

$$x = \frac{(1+\sqrt{2})^{2k} - (1-\sqrt{2})^{2k}}{2\sqrt{2}} + \frac{(1+\sqrt{2})^{2k-1} - (1-\sqrt{2})^{2k-1}}{2\sqrt{2}}$$

$$b = \frac{(1+\sqrt{2})^{2k} - (1-\sqrt{2})^{2k}}{4\sqrt{2}} + \frac{(1+\sqrt{2})^{2k-1} - (1-\sqrt{2})^{2k-1}}{4\sqrt{2}} - \frac{1}{2}$$

$$y = \frac{(1+\sqrt{2})^{2k} - (1-\sqrt{2})^{2k}}{2\sqrt{2}}$$

$$a = \frac{(1+\sqrt{2})^{2k} - (1-\sqrt{2})^{2k}}{4\sqrt{2}}$$

Therefore for any given k we can find the kth square triangular number. The a value will give us the square number required and the b value will give us the triangular number required.

For example with k = 3:

$$a = \frac{(1+\sqrt{2})^{2(3)} - (1-\sqrt{2})^{2(3)}}{4\sqrt{2}} = 35$$

$$b = \frac{(1+\sqrt{2})^{2(3)} - (1-\sqrt{2})^{2(3)}}{4\sqrt{2}} + \frac{(1+\sqrt{2})^{2(3)-1} - (1-\sqrt{2})^{2(3)-1}}{4\sqrt{2}} - \frac{1}{2} = 49$$

This tells us the 3rd square triangular number is the 35th square number or the 49th triangular number. Both these give us an answer of 1225 – which checking back from our table is the correct answer.

So, we have arrived at 2 possible methods for finding the square triangular numbers – one using modern computational power, and one using the skills of 18th century number theory.

3. Modelling and Calculus: Finding the volume of a rugby ball (prolate spheroid). [Suitable for SL students]

With the rugby union World Cup currently underway I thought I'd try and work out the volume of a rugby ball using some calculus. This method works similarly for American football and Australian rules football. The approach is to consider the rugby ball as an ellipse rotated 360 degrees around the x axis to create a volume of revolution. We can find the equation of an ellipse centered at (0,0) by simply looking at the x and y intercepts. An ellipse with y-intercept (0,b) and x intercept (a,0) will have equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Therefore for our rugby ball with a horizontal "radius" (vertex) of 14.2cm and a vertical "radius" (co-vertex) of 8.67cm will have equation:

$$\frac{x^2}{14.2^2} + \frac{y^2}{8.67^2} = 1$$

We can see that when we plot this ellipse we get an equation which very closely resembles our rugby ball shape:

Therefore we can now find the volume of revolution by using the following formula:

$$\text{Volume} = \pi \int_{-14.2}^{14.2} y^2 dx$$

But we can simplify matters by starting the rotation at x = 0 to find half the volume, before doubling our answer. Therefore:

$$\text{Volume} = 2\pi \int_{0}^{14.2} y^2 dx$$

Rearranging our equation of the ellipse formula we get:

$$y^2 = 8.67^2(1 - \frac{x^2}{14.2^2})$$

Therefore we have the following integration:

$$\text{Volume} = 2\pi(8.67)^2 \int_{0}^{14.2} (1 - \frac{x^2}{14.2^2})dx$$

$$\text{Volume} = 2\pi(8.67)^2 [x - \frac{x^3}{(3)14.2^2}]_{0}^{14.2}$$

$$\text{Volume} \approx 4471\ cm^3$$

Therefore our rugby ball has a volume of around 4.5 litres. We can compare this with the volume of a football (soccer ball) – which has a radius of around 10.5cm, therefore a volume of around 4800 cubic centimeters.

We can find the general volume of any rugby ball (mathematically defined as a prolate spheroid) by the following generalization:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2(1 - \frac{x^2}{a^2})$$

$$\text{Volume} = 2\pi(b)^2 \int_0^a (1 - \frac{x^2}{a^2})dx$$

$$\text{Volume} = 2\pi(b)^2[x - \frac{x^3}{(3)a^2}]_0^a$$

$$\text{Volume} = 2\pi(b)^2[a - \frac{a^3}{(3)a^2}]$$

$$\text{Volume} = 2\pi(b)^2[a - \frac{a}{(3)}]$$

$$\text{Volume} = \frac{4}{3}\pi(b)^2 a$$

We can see that this is very closely related to the formula for the volume of a sphere, which makes sense as the prolate spheroid behaves like a sphere deformed across its axes. Our prolate spheroid has "radii" b, b and a – therefore r cubed in the sphere formula becomes b squared a.

Prolate spheroids in nature

Prolate spheroids also appear in astronomy. The Crab Nebula which is a distant Supernova remnant around 6500 light years away can be described as a prolate spheroid. You could investigate other prolate spheroids and their models.

4. Trigonometry and modelling. Tides: What is the effect of a full moon? [Suitable for SL students]

Phuket Tide Tables 2018

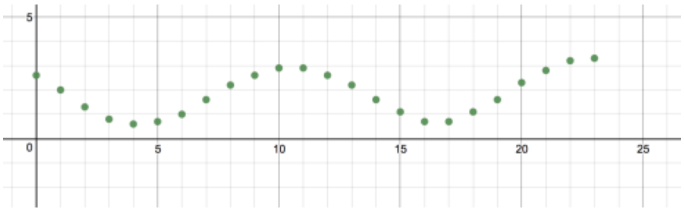
Phuket: **Sunday, 25 March 2018** 17:01:32

PREV | **March** | NEXT

Date	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1 ○	2.6	2.0	1.3	0.8	0.6	0.7	1.0	1.6	2.2	2.6	2.9	2.9	2.6	2.2	1.6	1.1	0.7	0.7	1.1	1.6	2.3	2.8	3.2	3.3
2	3.0	2.5	1.8	1.1	0.6	0.4	0.6	1.1	1.8	2.4	2.9	3.1	2.9	2.6	2.0	1.4	0.9	0.6	0.7	1.2	1.8	2.5	3.1	3.3
3	3.2	2.8	2.2	1.5	0.8	0.4	0.4	0.8	1.4	2.1	2.7	3.1	3.1	2.9	2.4	1.8	1.1	0.7	0.6	0.8	1.4	2.1	2.7	3.2
4	3.3	3.1	2.6	1.9	1.2	0.6	0.4	0.6	1.0	1.7	2.4	2.9	3.1	3.1	2.7	2.2	1.5	0.9	0.6	0.7	1.1	1.7	2.3	2.9
5	3.2	3.1	2.8	2.3	1.6	1.0	0.6	0.5	0.8	1.4	2.0	2.6	3.0	3.1	2.9	2.4	1.9	1.3	0.8	0.7	0.9	1.4	2.0	2.5
6	2.9	3.0	2.9	2.5	1.9	1.3	0.9	0.7	0.8	1.2	1.8	2.3	2.8	3.0	2.9	2.6	2.2	1.6	1.1	0.9	0.9	1.2	1.7	2.2
7	2.6	2.8	2.8	2.5	2.1	1.6	1.2	0.9	0.9	1.1	1.6	2.1	2.5	2.8	2.8	2.7	2.4	1.9	1.4	1.1	1.0	1.2	1.5	1.9
8	2.3	2.5	2.6	2.5	2.2	1.8	1.5	1.2	1.1	1.2	1.5	1.9	2.2	2.5	2.7	2.6	2.4	2.1	1.7	1.4	1.3	1.3	1.5	1.8
9 ☾	2.0	2.3	2.4	2.3	2.2	2.0	1.7	1.5	1.3	1.3	1.5	1.7	2.0	2.3	2.4	2.5	2.4	2.2	2.0	1.7	1.5	1.5	1.5	1.7
10	1.8	2.0	2.1	2.2	2.1	2.0	1.9	1.7	1.6	1.5	1.6	1.7	1.9	2.0	2.2	2.3	2.3	2.2	2.2	2.0	1.8	1.7	1.7	1.7
11	1.7	1.8	1.8	1.9	2.0	2.0	2.0	1.9	1.9	1.8	1.7	1.7	1.7	1.8	1.9	2.0	2.1	2.2	2.2	2.2	2.1	2.0	1.9	1.8
12	1.7	1.6	1.6	1.7	1.8	1.9	2.0	2.1	2.1	2.1	2.0	1.8	1.7	1.7	1.7	1.7	1.9	2.0	2.2	2.3	2.4	2.3	2.2	2.0
13	1.7	1.5	1.4	1.4	1.5	1.7	1.9	2.1	2.3	2.3	2.2	2.0	1.8	1.6	1.5	1.5	1.6	1.8	2.1	2.3	2.5	2.6	2.5	2.2
14	1.9	1.6	1.3	1.2	1.3	1.5	1.8	2.1	2.3	2.5	2.5	2.3	2.0	1.7	1.4	1.3	1.3	1.5	1.9	2.2	2.6	2.7	2.7	2.5
15	2.2	1.7	1.3	1.1	1.0	1.2	1.6	2.0	2.3	2.6	2.7	2.6	2.3	1.8	1.4	1.2	1.1	1.2	1.6	2.1	2.5	2.8	2.9	2.8
16 ●	2.5	2.0	1.4	1.0	0.8	0.9	1.3	1.8	2.2	2.6	2.8	2.8	2.5	2.1	1.6	1.2	0.9	1.0	1.3	1.8	2.3	2.8	3.0	3.0

Let's have a look at the effect of the moon on the tides in Phuket. The Phuket tide table above shows the height of the tide (meters) on given days in March, with the hours along the top. So if we choose March 1st (full moon) we get the following graph:

Phuket tide at full moon:



If I use the standard sine regression on Desmos I get the following:

$$y_1 \sim a \sin\left(b\left(x_1 + c\right)\right) + d$$

STATISTICS

$R^2 = 0.9848$

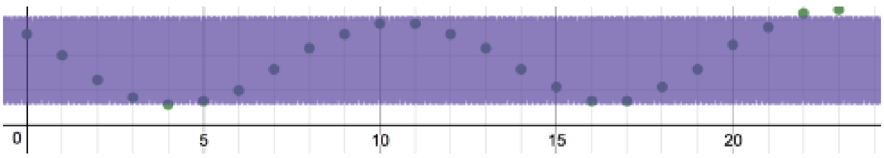
RESIDUALS

e_1 plot

PARAMETERS

$a = 1.22371$ $b = 119.898$

$c = -1.70868$ $d = 1.86145$



This doesn't look like a very useful graph – but the R squared value is very close to one – so what's gone wrong? Well, Desmos has done what we asked it to do – found a sine curve that goes through the points, it's just that it's chosen a b value of close to 120 – meaning that the curve has a very small period. So to prevent

Desmos doing this, we need to fix the period first. If we are in radians then we use the formula:

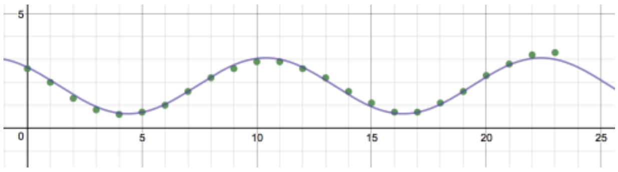
$$\text{period} = \frac{2\pi}{b}$$

Looking at the original graph we can see that this period is around 12. Therefore:

$$12 = \frac{2\pi}{b}$$

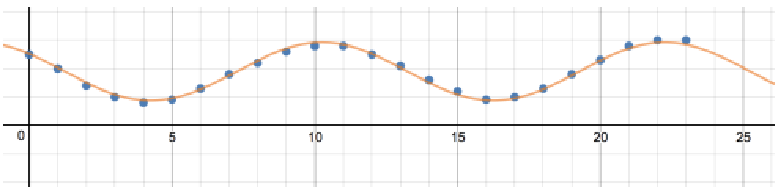
$$b = \frac{\pi}{6}$$

Now when I fix *b* and do a regression to find the other variables this gives something that looks a lot nicer:



Phuket tide at new moon:

Using the same approach I can now also fit a curve to describe the new moon tide:

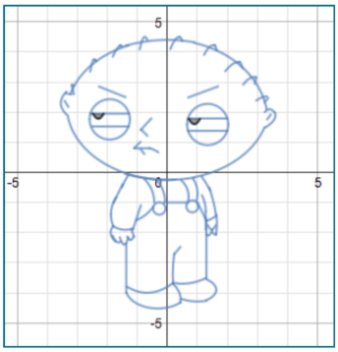


Both graphs show a very close fit to the original data – though both under-value the tide at 2300. We can see that the full moon has indeed had an effect on the amplitude of the sine curves – with the amplitude of 1.21m for the full moon and only 1.03m for the new moon.

Further study:

We could then see if this relationship holds throughout the year – is there a general formula to explain the moon's effect on the amplitude? We could also see how we have to modify the sine wave to capture the tidal height over an entire week or month. Can we capture it with a single equation (perhaps a damped sine wave?) or is it only possible as a piecewise function? We could also use some calculus to find the maximum and minimum points.

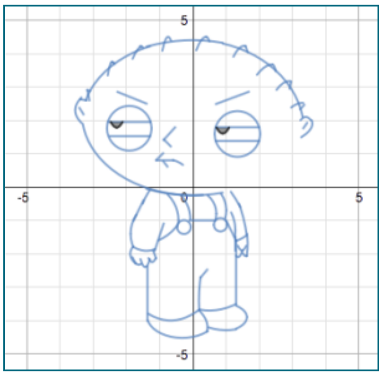
5. Modelling. Plotting Stewie Griffin from Family Guy [Suitable for SL students]



Computer aided design gets ever more important in jobs – and with graphing software we can create art using maths functions. For example the above graph was created by a user, Kara Blanchard on Desmos. You can see the original graph here, by clicking on each part of the function you can see which functions describe which parts of Stewie. There are a total of 83 functions involved in this picture. For example, the partial ellipse:

$$\frac{(x)^2}{11.4}+\frac{(y-2.2)^2}{6}=1$$

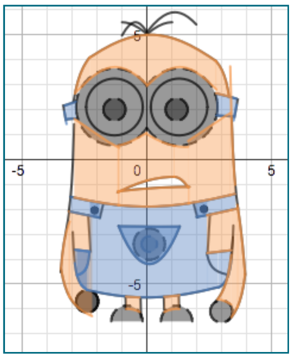
when x is bounded between 3.24 and 0.9, and y is bounded as less than 1.5 generates Stewie's left cheek. This is what he looks like without it:



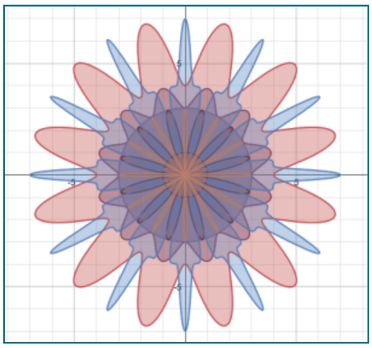
By clicking on the various functions you can discover which ones are required to complete the full drawing.

Other artwork designed by users includes:

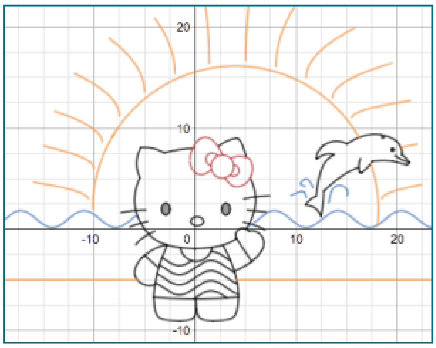
A minion from Despicable Me



A sunflower:



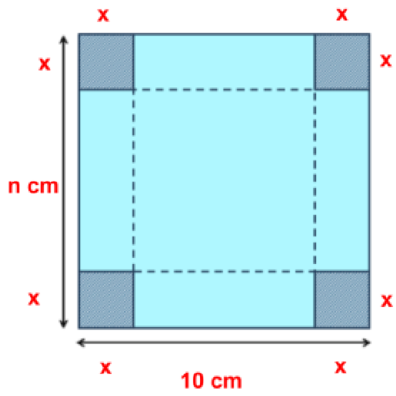
And Hello Kitty:



See if you can create some designs of your own! This could make an interesting maths investigation for anyone thinking about a career in computer design or art – as it is a field which will grow in importance in the coming years.

You might also like to look at a similar post on using Wolfram Alpha to plot the Batman and Superman logos.

6. Calculus optimisation. Volume optimization of a cuboid [Suitable for HL students]



This is an extension of the Nrich task which is currently live – where students have to find the maximum volume of a cuboid formed by cutting squares of size x from each corner of a 20 x 20 piece of paper. I'm going to use an n x 10 rectangle and see what the optimum x value is when n tends to infinity.

First we can find the volume of the cuboid:

$$\text{Volume} = x(10-2x)(n-2x)$$

$$\text{V} = 10xn - 20x^2 - 2x^2n + 4x^3$$

Next we want to find when the volume is a maximum, so differentiate and set this equal to 0.

$$\frac{dV}{dx} = 10n - 40x - 4xn + 12x^2$$

$$0 = 10n - 40x - 4xn + 12x^2$$

$$0 = 12x^2 - (40+4n)x + 10n$$

Next we use the quadratic formula to find the roots of the quadratic, and then see what happens as n tends to infinity (i.e we want to see what the optimum x values are for our cuboid when n approaches infinity). We only take the negative solution of the + – quadratic solutions because this will be the only one that fits the initial problem.

$$a = 12, \qquad b = -(40+4n), \qquad c = 10n$$

$$\frac{(40+4n) - \sqrt{1600 + 320n + 16n^2 - 4(12)(10n)}}{2(12)}$$

$$\frac{(40+4n) - \sqrt{1600 - 160n + 16n^2}}{2(12)}$$

$$\lim_{n\to\infty} \frac{(40+4n) - \sqrt{1600 - 160n + 16n^2}}{2(12)}$$

Next we try and simplify the square root by taking out a factor of 16, and then we complete the square for the term inside the square root (this will be useful next!)

$$\lim_{n\to\infty} \frac{(40+4n) - \sqrt{16}\sqrt{100 - 10n + n^2}}{2(12)}$$

$$\lim_{n\to\infty} \frac{(40+4n) - 4\sqrt{(n-5)^2 + 75}}{2(12)}$$

Next we make a u substitution. Note that this means that as n approaches infinity, u approaches 0.

$$u = \frac{\sqrt{75}}{(n-5)}$$

$$n = \frac{\sqrt{75}}{u} + 5$$

Substituting this into the expression gives us:

$$\lim_{u\to 0} \frac{\left(40 + 4(\frac{\sqrt{75}}{u} + 5)\right) - 4\sqrt{(\frac{\sqrt{75}}{u})^2 + 75}}{2(12)}$$

$$\lim_{u\to 0} \frac{\left(40 + 4\frac{\sqrt{75}}{u} + 20\right) - 4\sqrt{\frac{75}{u^2} + 75}}{2(12)}$$

We then manipulate the surd further to get it in the following form:

$$\lim_{u\to 0}\frac{\left(40+4\frac{\sqrt{75}}{u}+20\right)-4\sqrt{75}\sqrt{\frac{1}{u^2}+1}}{2(12)}$$

$$\lim_{u\to 0}\frac{\left(40+4\frac{\sqrt{75}}{u}+20\right)-4\sqrt{75}\frac{1}{u}\sqrt{1+u^2}}{2(12)}$$

Now, the reason for all that manipulation becomes apparent – we can use the binomial expansion for the square root of 1 + u^2 to get the following:

$$\lim_{u\to 0}\frac{\left(40+4\frac{\sqrt{75}}{u}+20\right)-4\sqrt{75}\frac{1}{u}(1+\frac{u^2}{2}+higher\ powers)}{2(12)}$$

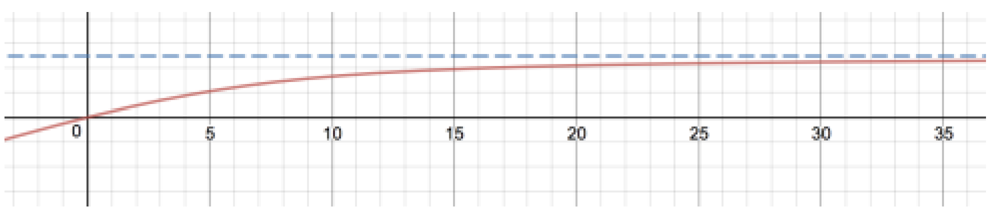
$$\lim_{u\to 0}\frac{\left(40+4\frac{\sqrt{75}}{u}+20\right)-4\sqrt{75}\frac{1}{u}-2\sqrt{75}u+higher\ powers}{2(12)}$$

$$\lim_{u\to 0}\frac{(40+20)}{2(12)}=2.5$$

Therefore we have shown that as the value of n approaches infinity, the value of x that gives the optimum volume approaches 2.5cm.

So, even though we start with a pretty simple optimization task, it quickly develops into some quite complicated mathematics. We could obviously have plotted the term in n to see what its behavior was as n approaches infinity, but it's nicer to prove it. So, let's check our result graphically.

$$x=\frac{(40+4n)-\sqrt{16}\sqrt{100-10n+n^2}}{2(12)}$$



As we can see from the graph, with n plotted on the x axis and x plotted on the y axis we approach x = 2.5 as n approaches infinity – as required.

An m by n rectangle.

So, we can then extend this by considering an n by m rectangle, where m is fixed and then n tends to infinity. As before the question is what is the value of x which gives the maximum volume as n tends to infinity?

We do the same method. First we write the equation for the volume and put it into the quadratic formula.

$$\text{Volume} = x(m-2x)(n-2x)$$

$$\text{V} = xmn - 2mx^2 - 2x^2n + 4x^3$$

$$\frac{dV}{dx} = mn - 4mx - 4xn + 12x^2$$

$$\frac{(4m+4n) - \sqrt{16m^2 + 16n^2 - 16mn}}{2(12)}$$

Next we complete the square, and make the u substitution:

$$\frac{(4m+4n) - 4\sqrt{(n-\frac{m}{2})^2 + \frac{3m^2}{4}}}{2(12)}$$

$$u = \frac{\sqrt{\frac{3m^2}{4}}}{(n-\frac{m}{2})}$$

$$n = \frac{\sqrt{\frac{3m^2}{4}}}{u} + \frac{m}{2}$$

Next we simplify the surd, and then use the expansion for the square root of 1 + u^2

$$\frac{\left(4m+4\left(\frac{\sqrt{\frac{3m^2}{4}}}{u}+\frac{m}{2}\right)\right)-4\sqrt{\frac{3m^2}{4}}\frac{1}{u}\sqrt{1+u^2}}{2(12)}$$

$$\lim_{u\to 0}\frac{\left(4m+4\left(\sqrt{\frac{3m^2}{4}}\frac{1}{u}+\frac{m}{2}\right)\right)-4\sqrt{\frac{3m^2}{4}}\frac{1}{u}\left(1+\frac{u^2}{2}+higher\ powers\right)}{2(12)}$$

This then gives the following answer:

$$=\frac{4m+\frac{4m}{2}}{24}$$

$$=\frac{m}{4}$$

So, we can see that for an n by m rectangle, as m is fixed and n tends to infinity, the value of x which gives the optimum volume tends to m divided by 4. For example when we had a 10 by n rectangle (i.e m = 10) we had x = 2.5. When we have a 20 by n rectangle we would have x = 5 etc.

And we've finished! See what other things you can explore with this problem.

IB Exploration Initial Submission Sheet:

Aim of your investigation:	
Outline of how your exploration could develop:	
Does this topic allow you to reflect on your results? How?	
What is the SL/HL maths (or equivalent) contained in this topic?	
Why did you choose this topic? What will be your personal engagement?	
What do you need to do next to get underway?	

Draft submission checklist:

1. Are you handing in what you believe to be a finished exploration?

2. Does your introduction set out a plan for what will happen in the investigation?

3. Does your introduction explain why you have chosen your topic and demonstrate a personal engagement?

4. Is your exploration easy to follow for a fellow IB student?

5. Do you label all graphs and define all mathematical terms? Do you use an equation editor throughout?

6. Do you have a bibliography and citations throughout?

7. Do you have a conclusion which discusses what you have learnt and discusses ideas for further study?

8. Have you reflected throughout the investigation as to the meaning of your results? What do they show? How could they be improved? Don't just arrive at results and move on!

9. Have you produced an individual piece of work? This should be personalised throughout.

10. Have you answered the question you set out to investigate? Can you reflect on how successful you have been? Have you altered your investigation to reflect your results?

Teacher marking

I thought I'd add some of my own thoughts with regards to teacher marking - other teachers may disagree, but this is what I have found helpful when moderating:

1. Teachers providing **evidence** for levels awarded on a cover-sheet. Just copying and pasting from the criteria points adds no extra value compared to just writing a level down.

Contrast:

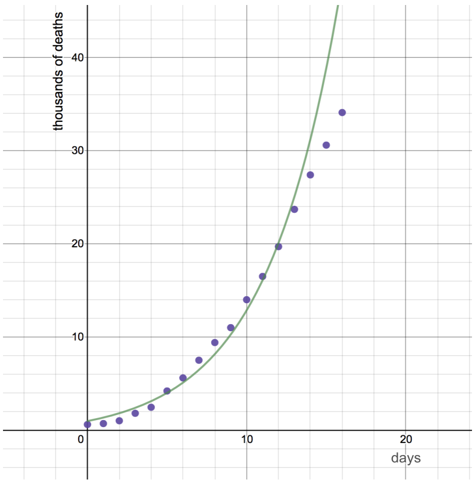
C2: There was meaningful engagement.

C2: The student is a very talented Grade 7 pianist who is passionate about music and so really was keen to do an exploration that allowed them to investigate the mathematics of music. I liked the personal element of trying to understand this topic from the perspective of how a computer would interpret music. This was a novel approach. The student did an interesting experiment to try to determine the pitch of a tuning fork and it was good to see cross-curricular work with the Physics department here . The student also did a lot of additional work on this topic to try to understand Fourier transforms – which required a lot of background reading. They showed real engagement in this topic throughout the whole process.

2. Teachers providing **annotation** on the work itself to highlight criteria evidence, to show that the mathematics has been checked (and is correct) or to highlight errors.

Example 1:

Figure 1: Plotting the number of deaths in a Covid-19 outbreak



B+ Axes are labelled, graph has a title. Domain restricted.

Example 2

I plotted the graph above and tried to fit an exponential curve to the data points. The graph needs to pass through (0,5) and (4,50), so I can make the following equations:

$5 = ae^{b(0)}$

$50 = ae^{b(4)}$

Solving these give:

A- Steps are not clearly explained

y = $5e^{0.576x}$

E+ Maths has been checked and is correct.

Example 3

I did linear regression and got an r value of 0.91. This shows there is a strong positive correlation between height and weight. As weight is my dependent variable this shows that as someone's height increases, their weight also increases. My linear regression line of best fit is:

$w = 10h + 40$

B+ Variables defined

w : *weight in kg*
h : *height in metres*

This means that for every extra metre in height we would expect someone to weigh an extra 10kg. However this equation is only valid for male adults, and because of the data we used the domain should be restricted to $h \geq 1.5$. We can see that without restricting the domain we would have a prediction of someone with no height being 40kg. This clearly makes no sense. I can use my equation to predict the weight of someone who is 1.7m tall:

$w = 10(1.7) + 40$
$w = 57$

D+ Meaningful attempt to reflect on real life intepretation

This seems like a reasonable prediction because I measured my classmate who is 1.7m tall and their weight was 63kg. This gives me confidence that my model is accurate.

Technology Guide

Desmos
Best for easy to use 2D graphing and regression.

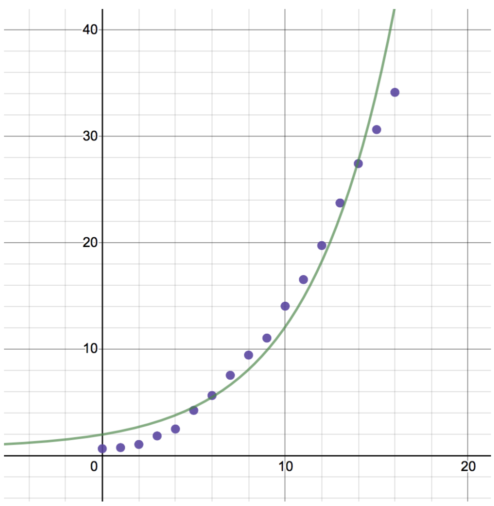
Regression method.

1) Enter a table of values (these can be pasted from another source such as Excel).

x_1	y_1
0	0.6
1	0.7
2	1
3	1.8

2) You can then find the relevant regression equation by using the following idea. (This will give an exponential regression equation).

$$y_1 \sim ae^{bx_1} + c$$

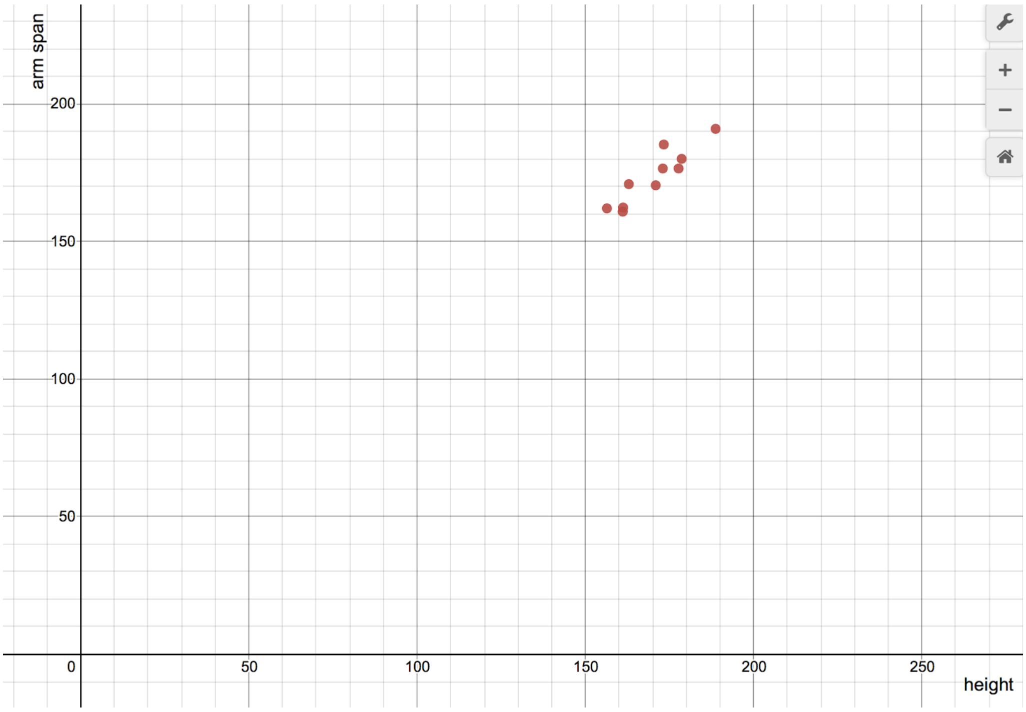


A quadratic regression would be given by:

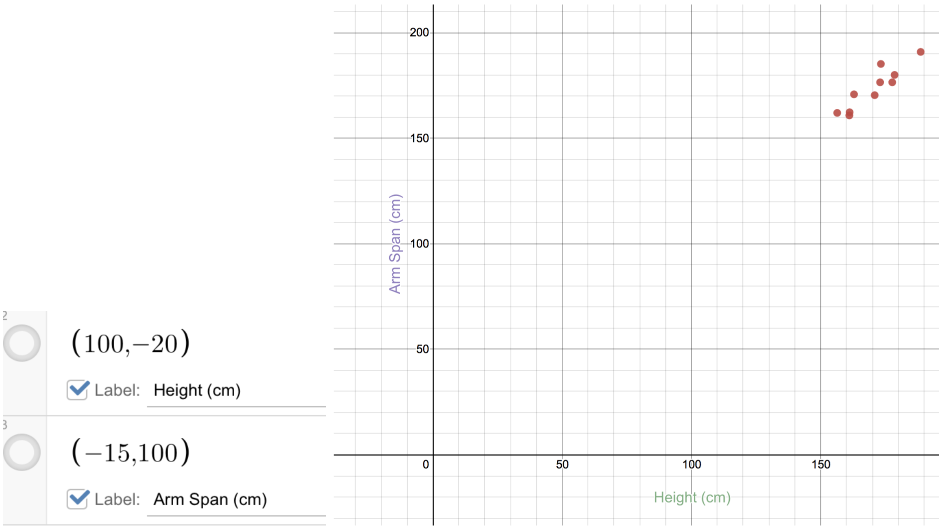
$$y_1 \sim ax_1{}^2 + bx_1 + c$$

Labelling axes in Desmos

The default labelling of axes in Desmos causes problems for students. Let's see what happens when we use the default method:



The labels do appear but are positioned such that we have to screen-capture to the edge of the graph window. One way around this is to label 2 points like this:

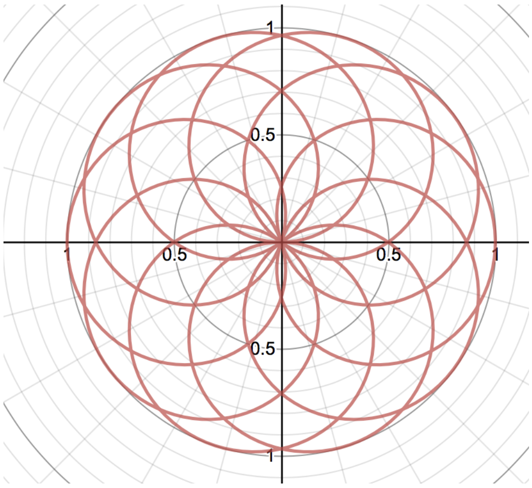


This allows us to position exactly where our axes labels will appear.

Interesting graphical features to explore on Desmos:

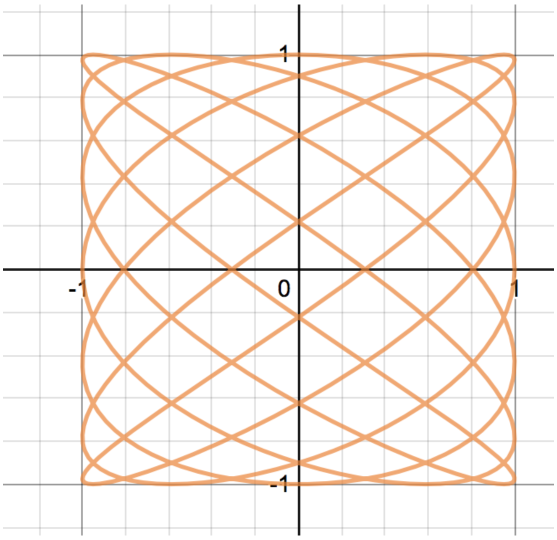
1) Polar coordinates. Desmos doesn't just handle Cartesian coordinates - it can also draw graphs using polar coordinates like the example below:

$$r = sin(\frac{a}{b}\theta)$$



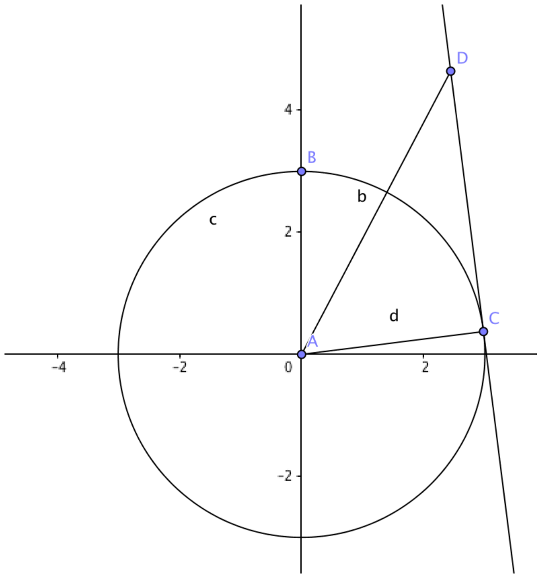
2) Parametric coordinates. Sometimes it's more convenient to plot the *x* and *y* coordinates in terms of a parameter. This can be done in desmos like this:

$$(sin(7\pi t),\ cos(7\pi t)\ ) \quad 0 < t < 2$$

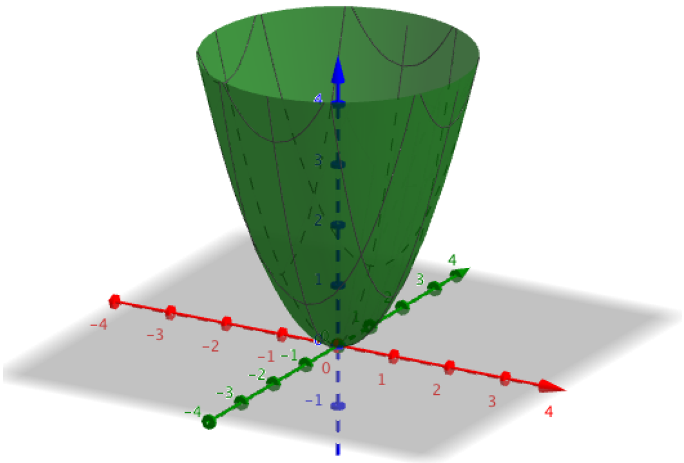


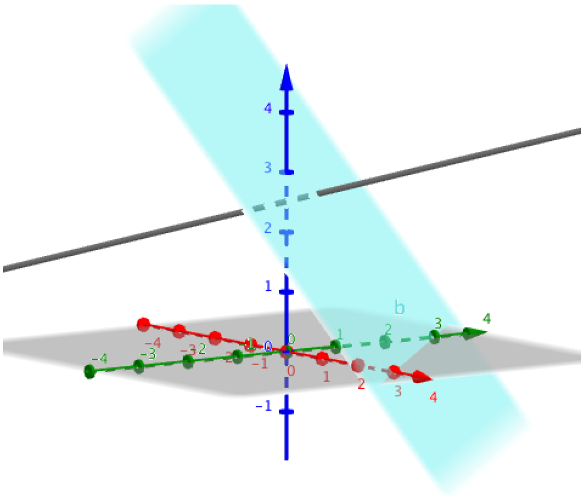
Geogebra
Best for 3D graphing and geometrical constructions.

Any work requiring geometric shapes can be done nicely using Geogebra. For example I can demonstrate one of the circle theorems below:



Geogebra is excellent for 3D graphics - which can fit well with vectors, volumes of revolution etc.

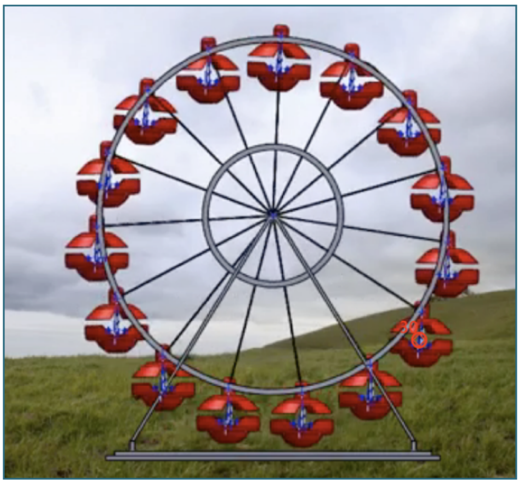




The first graph would make an interesting investigation into the volume of a cup, the second graph would allow a nice visual representation of the intersection of a line and a plane.

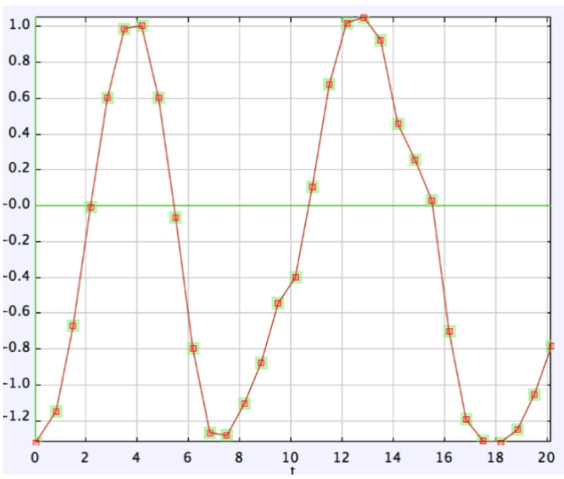
Tracker
Best for motion tracking investigation when you want to convert a video to a graph of motion.

It's a little bit difficult to get set up (but well worth it if this is the sort of investigation you plan to do). I'd recommend this excellent tutorial video here. The basic idea is that we can upload a video, then track a point that moves frame by frame (shift-ctrl-click).



For example, say I am interested in the circular motion of a Ferris wheel. I can choose to track many things (such as vertical height, horizontal height, vertical speed etc against time).

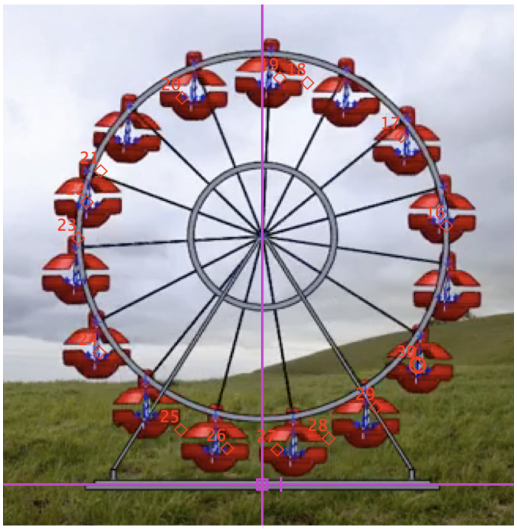
If I generate a graph of vertical height against time I can get the following graph:



Now Tracker will do some basic regression, but will also generate tables of values which can be copied and pasted in Desmos.

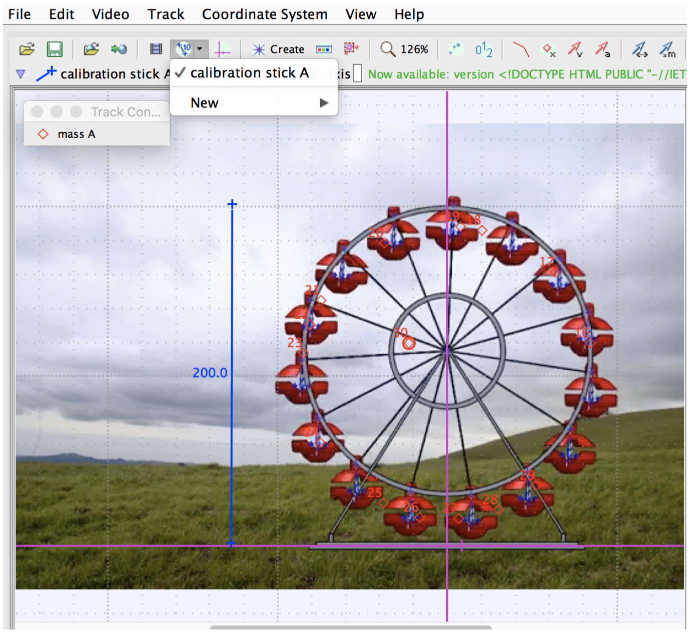
One problem you might notice with the graph above is that the scale is not correct - this is because I didn't tell Tracker the scale of the video that was uploaded. It assumes that your time is in seconds, but has no way of knowing what your distances are represented by.

Set your axes:



Make sure you set your axes such that the origin is where you want it to be.

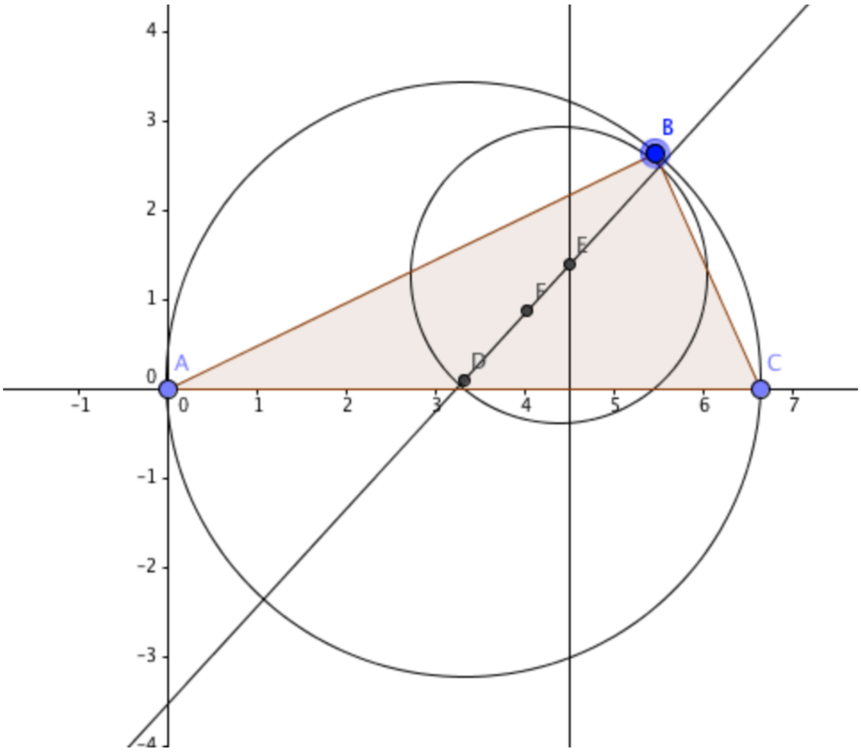
Use the calibration stick



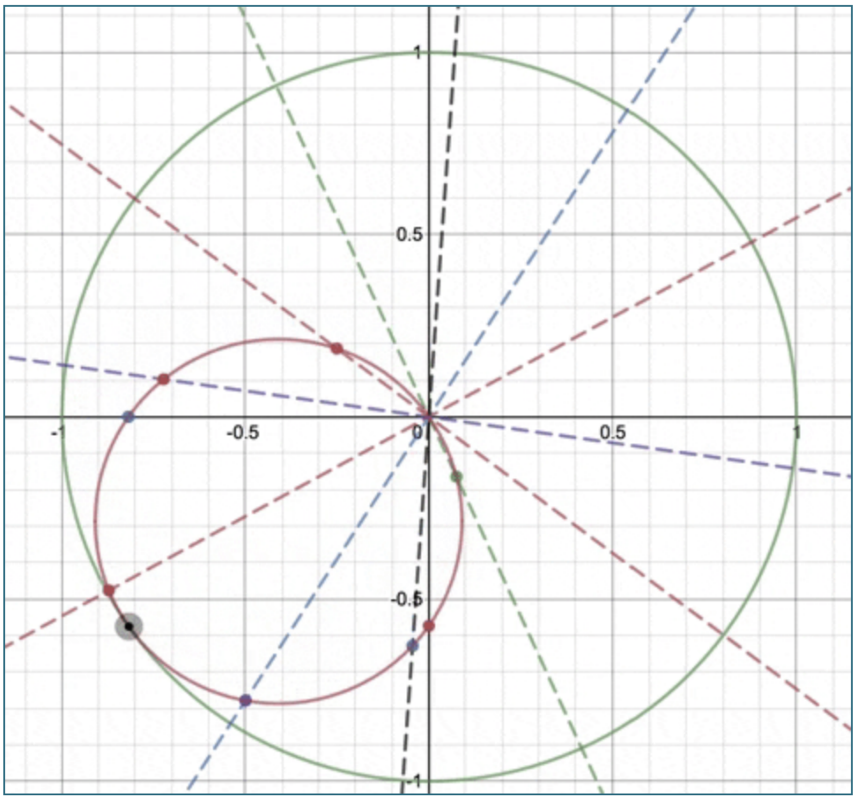
Now when I draw my graph, the origin will be where I want it to be, and the scale of the graph will be accurate.

Explorations with beautiful maths:

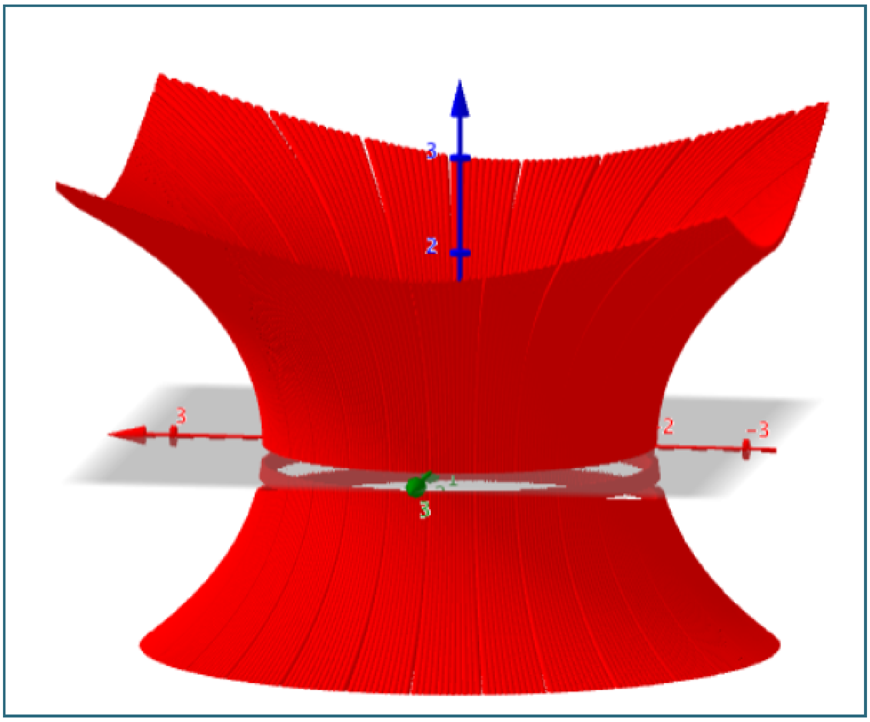
Euler's 9 point circle (using Geogebra)



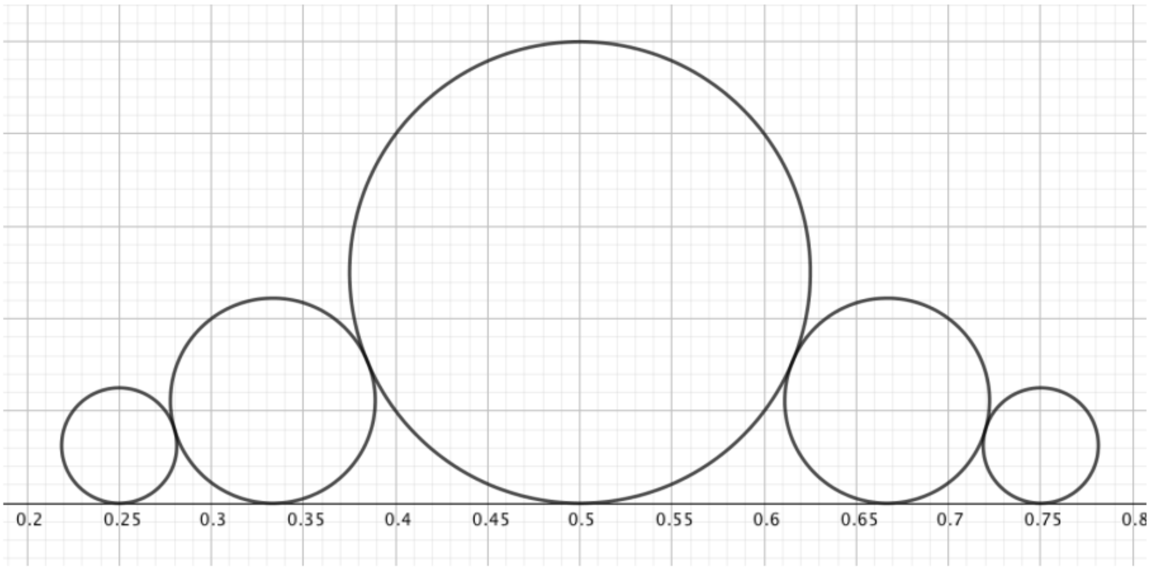
Tusi Couple (using Desmos)



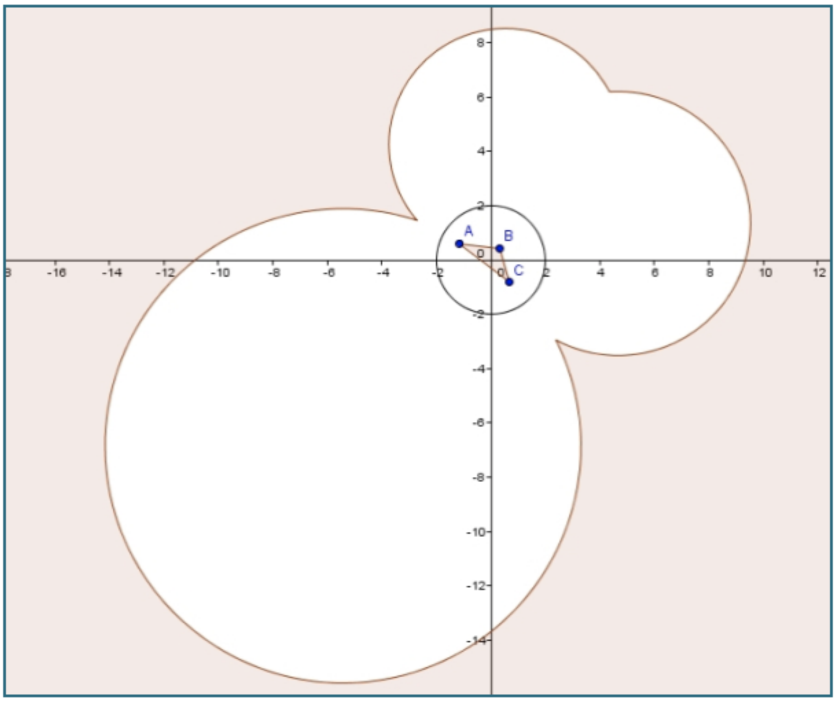
Catenoids (drawn using Geogebra 3D)



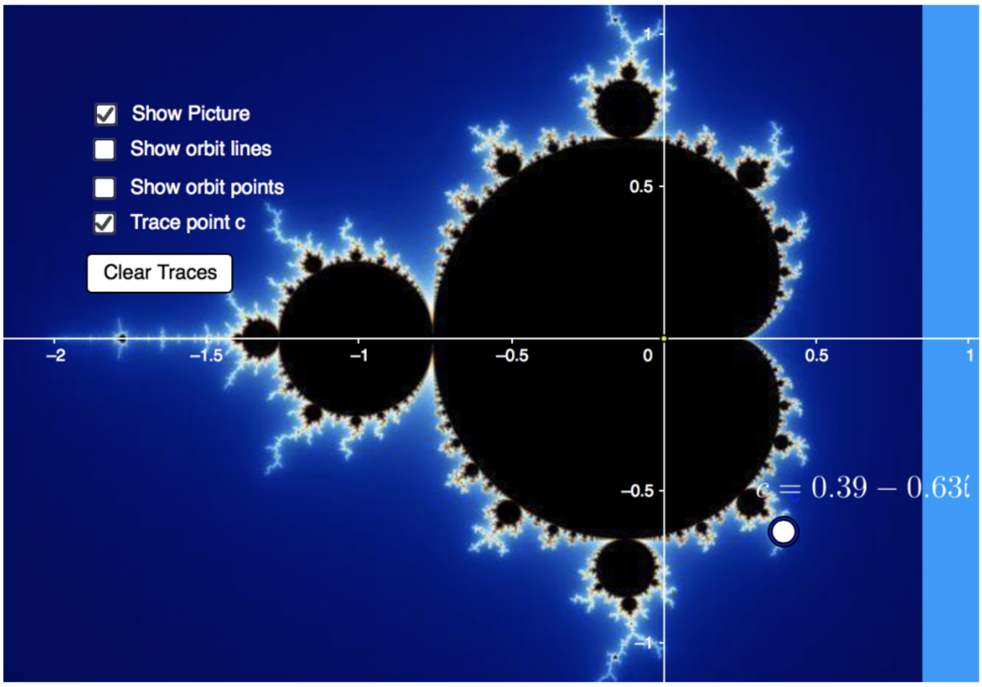
Ford Circles (drawn using Geogebra)



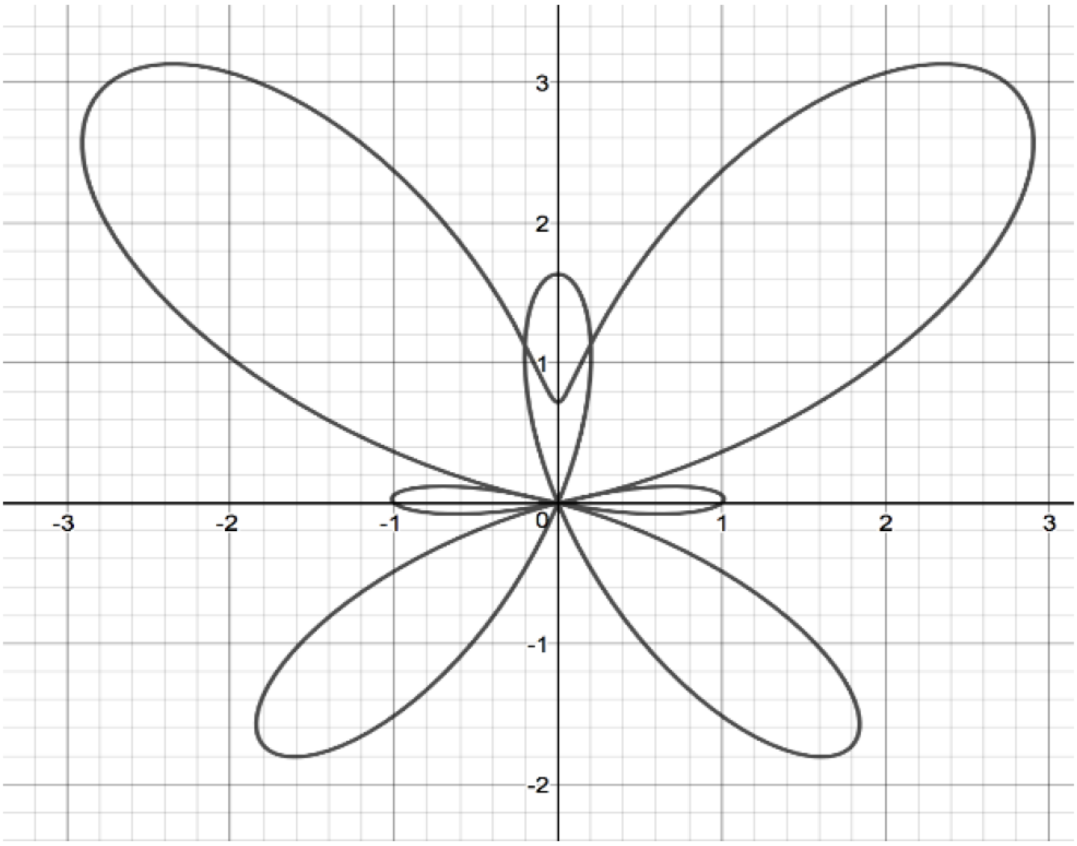
Circular inversions (drawn using Geogebra):



Plotting the Mandelbrot set (drawn using Geogebra):



Fun with polar coordinates (drawn with Desmos):



Torus (drawn with Geogebra):