Intermathematics
Maths Resources for International Schools

Phase portraits and exact solutions of coupled differential equations

(1) You are given the following differential equations.

$$\dot{x} = x - 2y$$

$$\dot{y} = x + dy$$

Show that this system of equations has eigenvalues:

$$\lambda = \frac{1 + d \pm \sqrt{d^2 - 2d - 7}}{2}$$

$$(1 - \lambda)(d - \lambda) + 2 = 0$$

$$\lambda^2 - (1 + d)\lambda + d + 2 = 0$$

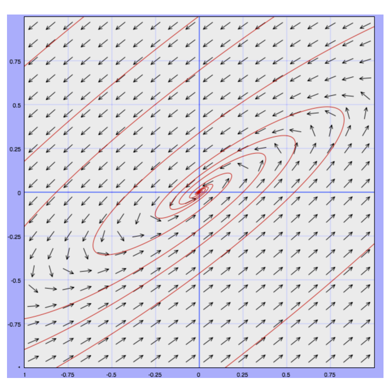
$$\lambda = \frac{1 + d \pm \sqrt{(1 + d)^2 - 4(d + 2)}}{2}$$

$$\lambda = \frac{1 + d \pm \sqrt{d^2 - 2d - 7}}{2}$$

(b) Hence describe the nature of the solutions to the coupled differential equations near the origin for the following values of d.

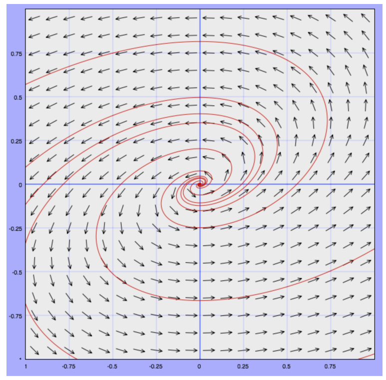
(i) $d = -1.5$

Complex with negative real part: solutions move towards the origin (spiral).



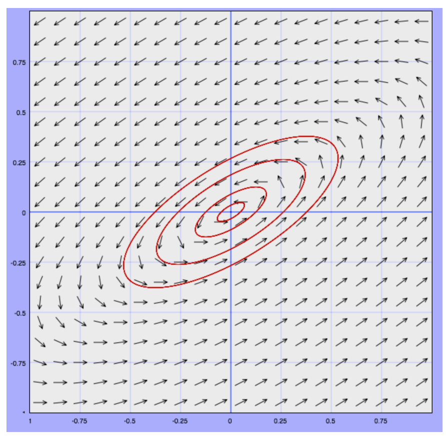
(ii) $d = 0$

Complex with positive real part: solutions move away from the origin (spiral).



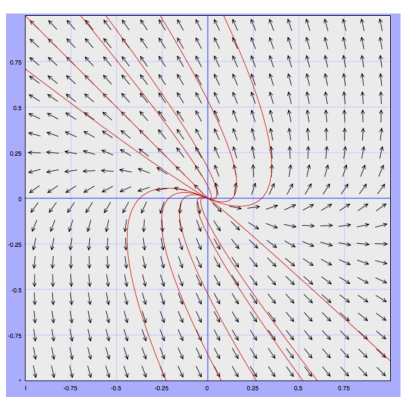
(iii) $d = -1$

Imaginary: solutions form a circle or ellipse



(iv) $d = 4$

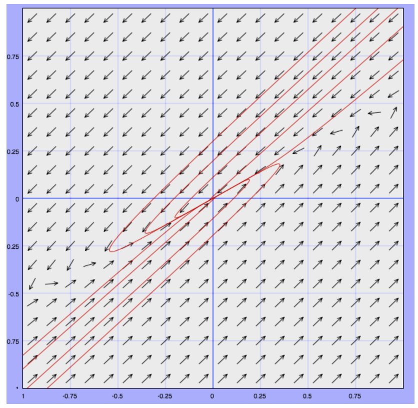
Real and positive: all solutions move away from origin (no spiral)



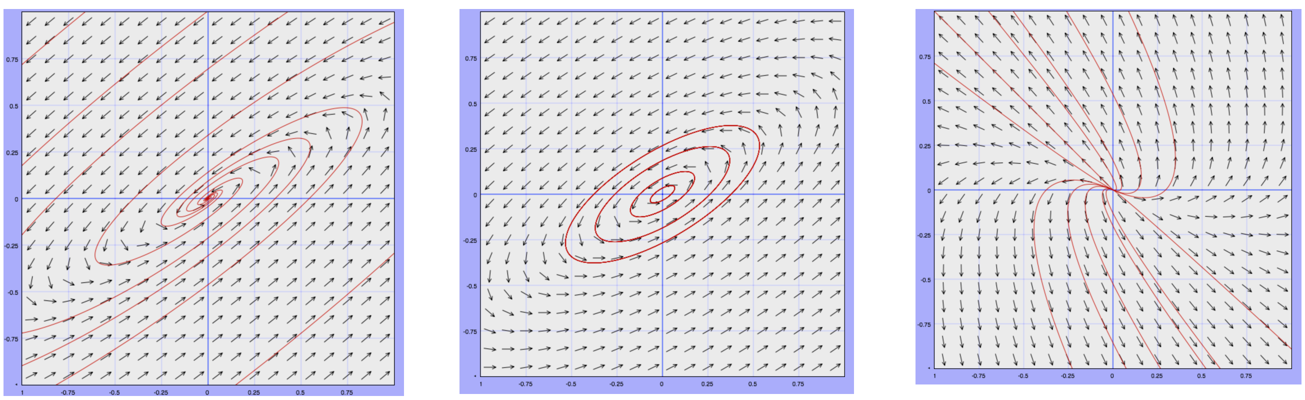
(v) $d = -1.9$

$d = -1.9$

Real and negative: solutions move towards the origin (no spiral)



(c) Which of the d values found above match the following phase portraits?



First phase portrait is $d = -1.5$

Second phase portrait is $d = -1$

Second phase portrait is $d = 4$

(2) You are given the following coupled differential equations:

$$\dot{x} = x + y$$

$$\dot{y} = 3x - y$$

(a) Find the eigenvalues and associated eigenvectors for these equations

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda = \pm 2$$

$\lambda = 2$

$$\begin{pmatrix} 1-2 & 1 \\ 3 & -1-2 \end{pmatrix}$$

$$-x + y = 0$$

$$3x - 3y = 0$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} 1+2 & 1 \\ 3 & -1+2 \end{pmatrix}$$

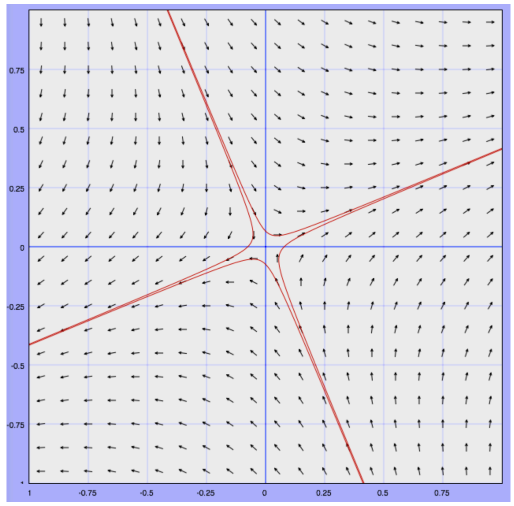
$$3x + y = 0$$

$$3x + y = 0$$

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

(b) Describe the behavior of the solutions near the origin.

The origin is a saddle:



(c) Find the exact solution given that $x = 1, y = 2$ when $t = 0$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = Ae^{0}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{0}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$1 = A + B$$

$$2 = A - 3B$$

$$B = -\frac{1}{4}$$

$$A = \frac{5}{4}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{5}{4}e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{4}e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

(3) You are given the following coupled differential equations:

$$\dot{x} = 3y$$

$$\dot{y} = -2x$$

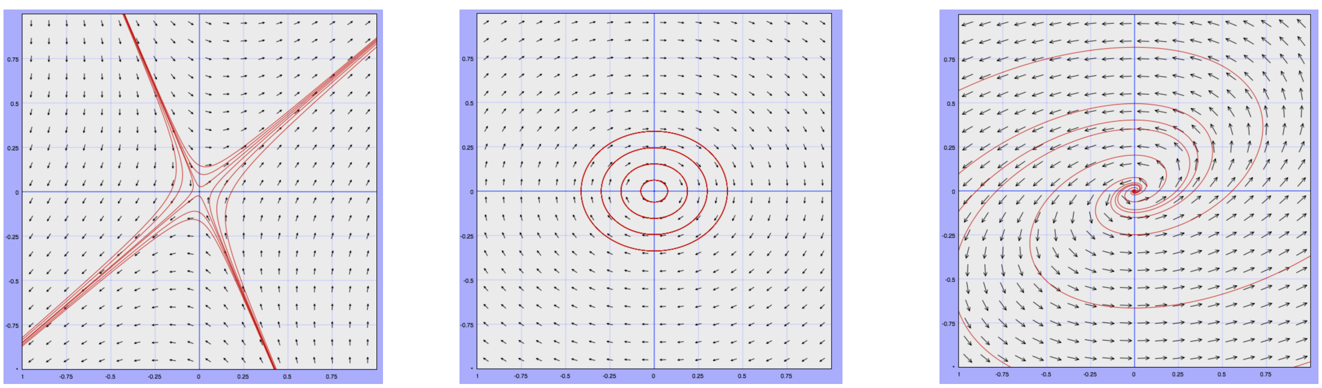
(a) Find the general solution to these equations.

$$\begin{vmatrix} 0-\lambda & 3 \\ -2 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 6 = 0$$

$$\lambda = \pm\sqrt{6}i$$

(b) Which of the following phase portraits represents this system?



The answer is the middle one (ellipse)

(4) You are given the following coupled differential equations:

$$\dot{x} = ax + by$$

$$\dot{y} = cx + dy$$

Show that these equations have distinct real eigenvalues when:

$$a^2 + b^2 > 2ad - 4bc$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$(a+d)^2 - 4(ad-bc) > 0$$

$$(a+d)^2 > 4(ad-bc)$$

$$a^2 + d^2 + 2ad > 4ad - 4bc$$

$$a^2 + d^2 > 2ad - 4bc$$