Vectors 4: Intersections of lines

Name..................................

(1) Find the vector equation of the line joining $A(2,3)$ and $B(-5,1)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

(b) Is the point $(16,7)$ on the line?

$$\begin{pmatrix} 16 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

no – no value of λ fits both coordinates.

(2) Two lines have vector equations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 12 \\ -8 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}$$

Show that the lines are skew.

$$2 + 2\lambda = 5$$
$$12\lambda = 2 - 6\mu$$
$$-1 - 8\lambda = -7 + 2\mu$$

Equation (1) gives

$$\lambda = \frac{3}{2}$$

Equation (2) gives

$$\mu = -\frac{8}{3}$$

But substitution into equation (3)

$$-1 - 8\times\left(\frac{3}{2}\right) \neq -7 + 2\times\left(-\frac{8}{3}\right)$$

Therefore skew.

(3) Two vector equations for the motion of two drones which set off at the same time are given below.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

Show that both drones will pass through the same point and find the coordinates of intersection. Will the drones crash into each other? Explain your answer.

$$5 + 2t_1 = -3 + 2t_2$$

$$3 + 2t_1 = 4 - t_2$$

$$1 + 2t_1 = 8 - 3t_2$$

Solving (1) and (2) gives

$$t_2 = 3, \qquad t_1 = -1$$

checking in (3) gives

$$1 - 2 = 8 - 9$$

So they both pass through the same coordinate:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$(3,1,-1)$$

But they don't crash because they pass through this point at different times (i.e $t_2 \neq t_1$).

(4) A car starts at the origin and is traveling in the direction $\binom{6}{8}$ with speed 40 m/s. In this question time t is in seconds and distances are in metres.

(a) Give a vector equation in the form:

$$\binom{x}{y} = \binom{a}{b} + t\binom{c}{d}$$

$$\frac{40}{10}\binom{6}{8} = \binom{24}{32}$$

$$\binom{x}{y} = \binom{0}{0} + t\binom{24}{32}$$

(b) A second car sets off at the same time with motion given by the vector equation:

$$\binom{x}{y} = \binom{12}{6} + t\binom{20}{30}$$

Will the two cars crash into each other? Explain your answer.

$$24t = 12 + 20t$$
$$32t = 6 + 30t$$

Both equations have solution $t = 3$.

They will both pass through the same coordinate (72, 96) after 3 seconds – and so will crash.

(5) Find the vector equation of the line L_1 between $A(1,5)$ and $B(3,14)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

(b) Find the shortest distance between the line L_1 and the point $C(10, 8)$ by considering the point of intersection.

Equation of the line through C, perpendicular to AB:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -9 \\ 2 \end{pmatrix}$$

Therefore the lines intersect when:

$$1 + 2\lambda = 10 - 9\mu$$

$$5 + 9\lambda = 8 + 2\mu$$

$$\lambda = \frac{9}{17}, \mu = \frac{15}{17}$$

Therefore the length of the line from C to AB is:

$$\left| \frac{15}{17} \begin{pmatrix} -9 \\ 2 \end{pmatrix} \right| = 8.13$$

(c) Find the shortest distance between the line L_1 and the point $C(10, 8)$ by considering graphing the distance to find a minimum.

General point on the line AB given by:

$$(1 + 2\lambda, 5 + 9\lambda)$$

Therefore the distance between AB and C given by:

$$\sqrt{(10 - (1 + 2\lambda))^2 + (8 - (5 + 9\lambda))^2}$$

This graph has a minimum when $\lambda = 0.529 \ldots$ with a distance of 8.13.

(4) t = 3