Sequence and Series 6: Mixed questions (HL)

(1) A 125.25 metre rope is cut into n pieces of increasing lengths that form an arithmetic sequence with a common difference of d metres. Given that the lengths of the shortest and longest pieces are 2 metres and 6.35 metres respectively, find the values of n and d.

$$S_n = \frac{n}{2}(2 + 6.35) = 125.25$$

$$n = 30$$

$$u_{30} = 2 + (29)(d) = 6.35$$

$$d = 0.15$$

(2) Find the sum of all three-digit natural numbers that are not exactly divisible by 5.

Sum of all 3 digit numbers:

$$100 + 101 + 102 + \cdots 999$$

$$S_{900} = \frac{900}{2}(100 + 999) = 494550$$

Sum of all 3 digit numbers divisible by 5:

$$100 + 105 + 110 + \cdots 995$$

$$S_{180} = \frac{180}{2}(100 + 995) = 98550$$

Therefore:

$$494550 - 98550 = 396000$$

(3) The mean of the first ten terms of an arithmetic sequence is -9.5. The mean of the first twenty terms of the arithmetic sequence is -24.5. Find the value of the 10^{th} term of the sequence.

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)d) = -95$$

$$S_{20} = \frac{20}{2}(2u_1 + (20-1)d) = -490$$

$$\frac{10}{2}(2u_1 + (10-1)d) = -95$$

$$2u_1 + 9d = -19$$

$$2u_1 + 19d = -49$$

$$d = -3$$

$$u_1 = 4$$

$$u_{10} = 4 + 9(-3) = -23$$

(4) An arithmetic sequence has first term a and has common difference d, $d \neq 0$. The 2^{nd}, 5^{th}, 14^{th} terms of the arithmetic sequence are the first three terms of a geometric sequence.

(a) Show that $a = 0.5d$.

$$a + d, a + 4d, a + 13d$$

$$\frac{a+4d}{a+d} = \frac{a+13d}{a+4d}$$

$$a^2 + 8ad + 16d^2 = a^2 + 14ad + 13d^2$$

$$3d^2 - 6ad = 0$$

$$d(3d - 6a) = 0$$

$$3d = 6a$$

$$a = 0.5d$$

(b) Given that the sum of the first 10 terms of the arithmetic sequence is 200, find the sum of the first 10 terms of the geometric sequence.

Arithmetic:

$$S_{10} = \frac{10}{2}(2(0.5d) + (10-1)(d)) = 200$$

$$10d = 40$$

$$d = 4$$

$$a = 2$$

Geometric:

$$u_1 = 6$$

$$r = \frac{0.5d + 4d}{0.5d + d} = 3$$

$$S_{10} = \frac{6((3)^{10} - 1)}{3 - 1} = 177144$$

(5) The common ratio of the terms in a geometric series is 3^x

(a) State the domain of values of x for which the sum to infinity of the series exists.

$$-1 < 3^x < 1$$

$$x < 0$$

(b) If the first term of the series is 60, find the value of x for which the sum to infinity is 80.

$$S_\infty = \frac{60}{1 - 3^x} = 80$$

$$x = \frac{\ln(0.25)}{\ln(3)} = -1.26$$

(6) A geometric sequence has a first term of 5 and a common ratio of 1.2. Find the number of terms needed such that the sum of the series is greater than 1000.

$$\frac{(5)((1.2)^n - 1)}{(1.2) - 1} > 1000$$

$$n > 20.4$$

$$n = 21$$

(7) On the first of January 2000, Fred decides to invest \$500. He invests in a bank offering 2% annual compound interest. On the first of January each following year he invests a further \$100.

(a) Show that on the first of January 2002 he will have \$722.20 in his account.

$$500 \times 1.02^2 + 100 \times 1.02^1 + 100 = 722.20$$

(b) How much money will be in his account on the first of January 2020?

$$500 \times 1.02^{20} + 100 \times 1.02^{19} + 100 \times 1.02^{18} + \cdots 100$$

$$500 \times 1.02^{20} + \frac{(100)((1.02)^{20} - 1)}{(1.02) - 1} = 3172.71$$

(8) A box contains 3 white balls and 1 black ball. Alice and Bob take it in turns to choose a ball. The ball is then replaced. The first person to choose a black ball wins. Alice goes first. What is the probability that Alice wins?

$$P(A\ wins) = \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} + \left(\frac{9}{16}\right)^2 \times \frac{1}{4} + \cdots$$

$$S_\infty = \frac{\frac{1}{4}}{1 - \frac{9}{16}} = \frac{4}{7}$$